The Invariant Complex Structure on the Homogeneous Space $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$

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Dipl.-Phys. Matthias Hofmann-Kliemt
aus Frankfurt am Main

Referent:	Prof. Dr. Karl-Hermann Neeb
Koreferent:	PD Dr. Thomas Püttmann
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Abstract in German

Sei $\mathsf{Diff}^+\mathbb{S}^1$ die Fréchet-Liegruppe aller orientierungserhaltenden Diffeomorphismen des Kreises $\mathbb{S}^1$ und $\mathsf{Rot}^+\mathbb{S}^1 \leq \mathsf{Diff}^+\mathbb{S}^1$ die Untergruppe der orientierungserhaltenden Isometrien bezüglich der Metrik auf $\mathbb{S}^1$. Wir betrachten den homogenen Raum $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$, auf dem die Gruppe $\mathsf{Diff}^+\mathbb{S}^1$ operiert, als eine glatte Fréchet-Mannigfaltigkeit. Das Hauptresultat dieser Arbeit ist, daß es auf $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ eine, bis auf ein Vorzeichen eindeutige, invariante komplexe Struktur gibt.

Um $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ mit einer komplexen Struktur zu versehen, benötigt man einen holomorphen Atlas. In dieser Arbeit wird ein solcher, aus genau einer Karte bestehender Atlas, konstruiert. Diese Konstruktion wird im Folgenden skizziert. Mit $\mathtt{V}^-$ bezeichnen wir die Menge aller diffeomorphen Einbettungen $f : \mathbb{S}^1 \to \mathbb{C}^\times$ mit Windungszahl

$$\int_{\mathbb{S}^1} \frac{f'(t)}{f(t)}\, dt = 1,$$

deren Fourier-Reihe die Form $f(t) = e^{it}(1 + a_1 e^{-it} + a_2 e^{-2it} + a_3 e^{-3it} + \cdots)$ mit $a_1, a_2, a_3, \ldots \in \mathbb{C}$ hat. Es hat sich herausgestellt, daß es genau einen Diffeomorphismus $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$ gibt, sodaß $f \circ \gamma$ eine Fourier-Reihe der Gestalt

$$(f \circ \gamma)(t) = e^{it}(r + b_1 e^{it} + b_2 e^{2it} + b_3 e^{3it} + \cdots)$$

besitzt, wobei $r \in \mathbb{R}^+$ und $b_1, b_2, b_3, \ldots \in \mathbb{C}$ sind. Auf diese Art und Weise wird eine Abbildung $\hat{\Gamma} : \mathtt{V}^- \to \mathsf{Diff}^+\mathbb{S}^1$ definiert. Wenn wir diese mit der kanonischen Projektion $\mathsf{Diff}^+\mathbb{S}^1 \to \mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1, \quad \gamma \mapsto \gamma \circ \mathsf{Rot}^+\mathbb{S}^1$ verketten, so erhalten wir eine Abbildung

$$\hat{\Psi} : \mathtt{V}^- \to \mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1,$$

für die wir zeigen, daß sie ein Diffeomorphismus ist. In dem Sinne, daß $\mathtt{V}^-$ eine offene Teilmenge eines affinen komplexen Unterraumes ist, haben wir $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ mit einer komplexen Struktur versehen. Weiterhin konnte gezeigt werden, daß diese komplexe Struktur invariant unter der Gruppenwirkung von $\mathsf{Diff}^+\mathbb{S}^1$ auf $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ ist und daß sie bis auf ein Vorzeichen die einzige invariante komplexe Struktur auf $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ ist.

Die Idee dieser Konstruktion geht auf Kirillov zurück, der sie in seiner Arbeit [16] für die Gruppe aller orientierungserhaltenden analytischen Diffeomorphismen des Kreises $\mathbb{S}^1$ durchführt und auf Stetigkeits- und Glattheitsbetrachtungen ganz verzichtet.

Chapter 1

Introduction

Motivation

Infinite-dimensional Lie groups and their representations arise in areas with symmetries generated by an infinite number of parameters. A famous example is general relativity and the approaches to its quantization [27]. Here very little experimental data are to be expected in the near future. Hence, to make any predictions, without the taste of arbitrariness and randomness, a rigorous treatment of the mathematical methods is unavoidable.

For this purpose, the rigorous theory of diffeomorphism groups and their representations proves to be a powerful tool. The most explored diffeomorphism group is $\mathsf{Diff}^+\mathbb{S}^1$, the group of diffeomorphisms on the circle $\mathbb{S}^1$. The founders of string-theory (see e.g. [30]) approached this topic and constructed unitary representations of the Virasoro–Lie algebra. The Virasoro–Lie algebra[1] is the central extension of the Lie algebra of vector fields $\mathsf{Vec}(\mathbb{S}^1)$, which is the Lie algebra of $\mathsf{Diff}^+\mathbb{S}^1$. The corresponding group representations have been established by Goodman and Wallach [12].

It turns out that the complex Virasoro–Lie algebra admits a root decomposition in the same manner as the finite-dimensional simple Lie algebras do. From this point of view, it is natural to investigate what can be carried over from the finite-dimensional case of simple Lie algebras to the diffeomorphism group $\mathsf{Diff}^+\mathbb{S}^1$ and to its central extension, the Virasoro group, respectively. One observation in the finite-dimensional case is that the homogeneous space G/H admits an invariant complex structure[2] if, for example, G is a simple compact real Lie group and $H \leq G$ is a max-

[1]Here, we consider the Virasoro–Lie algebra as a Fréchet-Lie algebra with the usual topological closure.

[2]This complex structure is determined by the root decomposition of the corresponding Lie algebra.

imal torus. The most common example of such a homogeneous space is the compact space $\mathsf{SU}(2)/\mathsf{U}(1)$, which is isomorphic to the Riemann sphere considered as a complex manifold. Another example is the homogeneous space $\mathsf{SU}(1,1)/\mathsf{U}(1)$ which is isomorphic to the open unit disk $\mathbb{D}$ in the complex plane. Like the group $\mathsf{SU}(1,1)$, the latter homogeneous space is non-compact. For the Heisenberg group, the corresponding complex manifold is the complex plane itself.

One can naturally ask if the corresponding homogeneous space for the Virasoro group, respectively, for $\mathsf{Diff}^+\mathbb{S}^1$, admits an invariant complex structure. It is not hard to show that an integrable almost complex structure exists on the homogeneous space $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$. In the finite-dimensional case the Newlander-Nirenberg Theorem [32, 24] asserts that every integrable almost complex structure is a complex structure. But since the Frobenius Theorem fails for Fréchet manifolds, it is by no means obvious, how to extend the proof of the Newlander-Nirenberg Theorem to $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$.

The idea, how to solve this problem, is attributed to Kirillov [16] and was the inspiring spark to seek for a rigorous proof. This leads to the main result of this thesis, which will be presented in the following.

The Main Result of this Thesis

Consider the 1-sphere $\mathbb{S} := \mathbb{R}/2\pi\mathbb{Z}$ as a Riemann manifold. Let $\mathsf{Diff}^+\mathbb{S}^1$ be the group of all orientation preserving diffeomorphisms, endowed with its Fréchet topology, and $\mathsf{Rot}^+\mathbb{S}^1$ be the subgroup of all orientation preserving isometries. The main result of the thesis is that on the homogeneous space $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$, there exists exactly one complex structure (up to sign) which is invariant under the action of $\mathsf{Diff}^+\mathbb{S}^1$.

In the following, we sketch the way to this result. The original idea of the construction goes back to Kirillov [16]. Albeit most of his constructions work for the group of analytic diffeomorphisms and can be understood with basic complex analysis and a good geometric imagination, the extension to the smooth case is much harder and requires more sophisticated techniques.

Let $\mathtt{V} \subseteq \mathcal{C}^\infty(\mathbb{S}^1)_\mathbb{C}$ be the subset of diffeomorphic embeddings $f : \mathbb{S}^1 \to \mathbb{C}^\times$ with winding number $\mathtt{w}(f) = 1$, let $\mathtt{V}^E \subseteq \mathtt{V}$ be the subset of functions of the form

$$f(t) = e^{it}(r + a_1 e^{it} + a_2 e^{2it} + \cdots)$$

with Fourier coefficients $r \in \mathbb{R}^+$ and $a_1, a_2, \ldots \in \mathbb{C}$, and $\mathtt{V}^+ \subseteq \mathtt{V}^E$ be the subset with $r = 1$. Moreover, let $\mathtt{V}^- \subseteq \mathtt{V}$ be the subset of functions with Fourier decomposition of the form

$$f(t) = e^{it}(1 + a_1 e^{-it} + a_2 e^{-2it} + \cdots)$$

with $a_1, a_2, \ldots \in \mathbb{C}$. This is an open subset of an affine subspace of a complex Fréchet space. We show the existence of the complex structure on $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ by constructing a smooth chart $\psi : \mathtt{V}^- \to \mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$. Now we say a few words about the construction of ψ.

The main result in Chapter 4 is that the composition map

$$\mathtt{C} : \mathtt{V}^E \times \mathsf{Diff}^+\mathbb{S}^1 \to \mathtt{V}, \quad (f, \gamma) \mapsto f \circ \gamma$$

is bijective. To prove this, we need the Riemann Mapping Theorem and a particular result [3] about the boundary behavior. We write

$$\hat{\Lambda} : \mathtt{V} \to \mathtt{V}^E, \quad f \mapsto \hat{\Lambda}(f) = \pi_1 \circ \mathtt{C}^{-1}$$

for the left component of $\mathtt{C}^{-1}$ and

$$\hat{\Gamma} : \mathtt{V} \to \mathsf{Diff}^+\mathbb{S}^1, \quad f \mapsto \hat{\Gamma}(f) = \pi_2 \circ \mathtt{C}^{-1}$$

for the right component, where π_1 and π_2 denote the projections onto the left and right component, respectively.

Hence, we can define the chart by

$$\psi : \mathtt{V}^- \to \mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1, \quad g \mapsto \hat{\Gamma}(g)^{-1} \circ \mathsf{Rot}^+\mathbb{S}^1.$$

The next step (Theorem 8.3.2) is to show that the chart ψ is bijective. To do this, we need the theorem that is the result of Chapter 3, and we call it *the Birkhoff decomposition* (3.4.1):
For each $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$ there exist two functions $f \in \mathtt{V}^+$, $g \in \mathtt{V}^-$, and a complex number $c \in \mathbb{C}^\times$ such that

$$f(\gamma(t)) = c \cdot g(t)$$

for all $t \in \mathbb{S}^1$. Furthermore, g, f and c are unique. Now we say some words about the proof. To show the existence of f and g, we proceed as follows. We build up a series $(\Gamma_N)_{N \in \mathbb{N}}$ of smooth auxiliary functions $\Gamma_N : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that $\Gamma_N(e^{it}) = e^{it}$ holds for every $N \in \mathbb{N}$, and

$$\mu_N : \overline{\mathbb{C}} \to \mathbb{C}, \quad z \to \begin{cases} \frac{\partial_{\bar{z}} \Gamma_N(z)}{\partial_z \Gamma_N(z)} & \text{for} \quad z \in \mathbb{D} \\ 0 & \text{for} \quad z \in \overline{\mathbb{C}} \backslash \mathbb{D} \end{cases}$$

is $(N+5)$-times continuously differentiable. Here $\overline{\mathbb{C}}$ is the Riemann sphere. With results concerning elliptic partial equations [10], we show that for every $N \in \mathbb{N}$, the solution $\alpha_N : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the Beltrami equation

$$\partial_{\bar{z}} \alpha_N(z) = \mu_N(z) \cdot \partial_z \alpha_N(z)$$

which satisfies $\alpha(0)$, $\alpha'(0) = 0$, and $\alpha(\infty) = \infty$, is N-times continuously differentiable. The existence of such a solution is well-known [1], and is a quasiconformal map. It turns out that $f \in \mathtt{V}^+$ and $g \in \mathtt{V}^-$, defined by

$$\begin{aligned} f(t) &:= \alpha(e^{i\gamma^{-1}(t)}), \\ g(t) &:= \underbrace{\frac{1}{\alpha'(\infty)}}_{c} \alpha(e^{-it}), \end{aligned}$$

are independent from N, and therefore smooth.

The next step is to show that ψ is a diffeomorphism. For this purpose, we introduce in Chapter 5 the concept of tameness on Fréchet spaces [13]. Tame Fréchet spaces are Fréchet spaces with an additional structure: a grading of their semi-norms. Based on this grading, the Nash-Moser Theorem provides an Inverse Function Theorem for tame Fréchet spaces, and we can prove in Chapter 7 that the composition map

$$\mathtt{C} : \mathtt{V}^E \times \mathsf{Diff}^+\mathbb{S}^1 \to \mathtt{V}, \quad (f, \gamma) \mapsto f \circ \gamma$$

is a diffeomorphism. In Chapter 8, we use the concept of tameness to prove that $\psi : \mathtt{V}^- \to \mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ is a diffeomorphism. The chart ψ induces a complex structure on $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$.

Moreover, in Chapter 8, we show that the induced complex structure is invariant, and that there exists no other invariant complex structure on $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ up to sign.

Further Aspects

A very remarkable aspect of the set $\mathtt{V}^-$ is the following. From Corollary 8.6.9, we obtain that we can use $\mathtt{V}^+$ instead of $\mathtt{V}^-$, and applying techniques provided by Chapter 2, it follows that $\mathtt{V}^+$ as well as the homogeneous space $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ can be identified with the subset $S \subseteq \mathsf{Hol}(\mathbb{D}, \mathbb{C})$ of univalent functions $F : \mathbb{D} \to \mathbb{C}$ which have diffeomorphic extensions to the boundary and satisfy $F(0) = 0$ and $F'(0) = 1$. The Fourier series of such a function is

$$F(z) = z(1 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots).$$

The Bieberbach–de Branges Theorem [7], in former times known as the Bieberbach conjecture, asserts that

$$|a_n| \leq n$$

holds for all $n > 1$. Thus S, respectively $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$, is a sort of infinite-dimensional bounded domain. This point of view was picked up by Hong and Rajeev [14], who computed explicitly a homogeneous Kähler metric on S, respectively $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$. Their strategy was to embed S holomorphically into the Segal[3] disk. This Kähler metric on $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ might enable us to construct unitary representations of the Virasoro group, respectively, projective unitary representations of $\mathsf{Diff}^+\mathbb{S}^1$, as suggested by Kirillov [17, 15]. In the terms of geometric quantization [34], this means that the Kähler manifold $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ corresponds to a symplectic manifold with a holomorphic complex polarization.

An alternative approach to gain an invariant complex structure on

$$\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$$

is the one of Nag and Verjovsky [22]. Consider the complex Banach space $L^\infty(\mathbb{D},\mathbb{C})$ of integrable functions (actually the elements are classes of functions). Let $B_1 \subseteq L^\infty(\mathbb{D},\mathbb{C})$ the open unit ball. We say two elements $\mu,\nu \in B_1$ are equivalent denoted by $\mu \sim \nu$ if there exist two quasi-conformal functions $f,g : \overline{\mathbb{D}} \to \mathbb{D}$ such that $f_{\bar z} = \mu\, f_z$ and $g_{\bar z} = \nu\, g_z$ hold on the interior $\mathbb{D}$ and that f coincides with g on the boundary $\partial\mathbb{D}$. The factor set $B_1/\sim$ is the universal Teichmüller space. A diffeomorphism $\gamma \in \mathsf{Diff}^+(\partial\mathbb{D})$ has a quasi-conformal extention $\tilde\gamma : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that $g_{\bar z}/g_z \in B_1$. This defines a function $\mathsf{Diff}^+(\partial\mathbb{D}) \to B_1/\sim$ and hence a function

$$\mathsf{Diff}^+(\partial\mathbb{D})/\mathsf{SU}(1,1) \to T(1) = B_1/\sim$$

into the Teichmüller space. It turns out that the induced complex structure on the homogeneous space $\mathsf{Diff}^+(\partial\mathbb{D})/\mathsf{SU}(1,1)$ is invariant under the action of $\mathsf{Diff}^+\mathbb{S}^1$. The homogeneous space $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ is treated as a holomorphic fiber space over $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{SL}(2,\mathbb{R})$.

Lempert endows the Virasoro group with the structure of a complex manifold [18]. This complex manifold is considered as a holomorphic line bundle over $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$.

Moreover, it should be mentioned that Edward Witten [33] discusses the homogeneous spaces $\mathsf{Diff}^+\mathbb{S}^1/\mathbb{S}^1$ and $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{SU}(1,1)$ as coadjoint orbits of the Virasoro group.

[3]The Segal disk is named after Graeme Segal [31] and is an infinite dimensional analog of the well known Siegel disk.

About the last chapter

The last chapter of this thesis is independent of the other chapters, and the following two assertions are proved.

Consider an irreducible representation $\pi : A \to \mathsf{B}(H)$ of a C^*-algebra A on a Hilbert space H. Fix a natural number $n \in \mathbb{N}$. The first result (9.3.4) is that the unitary group of A acts transitively on the Grassmannian $Gr_n(H)$, which is the set of all n-dimensional subspaces of H.

The second result of this chapter is the following. Let H_0 be a Hilbert space, and $\mathsf{U}_K(H_0)$ be the group of all unitary operators u such that $u - \mathbb{1}$ is compact. Moreover, let $\pi : \mathsf{U}_K(H_0) \to \mathsf{U}(V)$ be a norm-continuous unitary representation on a Hilbert space V. Then the image of a coherent state vector $w \in V$ with $w \neq 0$ under the momentum map has the form

$$\phi([w]) = \frac{1}{i} \sum_{k \in I} c_k |a_k\rangle\langle a_k|,$$

where $\{a_k\}_{k \in I} \subseteq V$ is an orthonormal basis, $c_j \in \mathbb{Z}$, and only finitely many coefficients are non-zero.

Chapter 2

Basic Concepts

In this chapter, we present basic concepts, which are used throughout the whole thesis and are necessary to formulate the results. In Section 1 we introduce the diffeomorphism group $\mathsf{Diff}^+\mathbb{S}^1$. Section 2 concerns the Fourier series of 2π-periodic functions. Section 3 deals with holomorphic functions on the unit disk $\mathbb{D}$ and their relation with the Fourier series of 2π-periodic functions. In Section 4, we introduce winding numbers, so that we are able to define the subsets $\mathtt{V}, \mathtt{V}^+, \mathtt{V}^-, \mathtt{V}^E \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$. These are the key objects on the way to prove the main result (8.6.11) about the complex structure on the homogeneous space $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$.

2.1 Diffeomorphism Group $\mathsf{Diff}^+\mathbb{S}^1$

In this section, we introduce the diffeomorphism group $\mathsf{Diff}^+\mathbb{S}^1$ as a factor group of its universal covering group $\widetilde{\mathsf{Diff}}^+\mathbb{S}^1$. The elements of $\mathsf{Diff}^+\mathbb{S}^1$ are thus equivalence classes of real-valued functions.

Definition 2.1.1 (1-sphere)**.** We define the sphere $\mathbb{S}^1$ to be the factor set

$$\mathbb{S}^1 := \mathbb{R}/2\pi\mathbb{Z} = \{\, t + 2\pi\mathbb{Z} \subseteq \mathbb{R} \,:\, t \in \mathbb{R} \,\}.$$

In this sense a function $f : \mathbb{S}^1 \to A$ from the sphere $\mathbb{S}^1$ into an arbitrary set A is the same as a 2π-periodic function $f : \mathbb{R} \to A$.
Remark: If we write

$$\int_{\mathbb{S}^1} f(t)dt$$

for a 2π-periodic function f, we take the Lebesgue integral over an arbitrary interval of length 2π, e.g., $\int_0^{2\pi} f(t)dt$.

Definition 2.1.2. We define $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ to be the complex vector space of all smooth 2π-periodic functions $f : \mathbb{S}^1 \to \mathbb{C}$ into the complex numbers. Similarly, we write $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$ for the real vector space of all smooth real valued functions $f : \mathbb{S}^1 \to \mathbb{R}$.

Definition 2.1.3. We define $\widetilde{\mathsf{Diff}}^+\mathbb{S}^1$ to be the set of all smooth functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that
(i) $f(t + 2\pi) = f(t) + 2\pi$ (*equivariance*) and
(ii) $f'(t) > 0$ hold.

Lemma 2.1.4. *On $\widetilde{\mathsf{Diff}}^+\mathbb{S}^1$ the composition map*

$$\mathsf{C} : \widetilde{\mathsf{Diff}}^+\mathbb{S}^1 \times \widetilde{\mathsf{Diff}}^+\mathbb{S}^1 \to \widetilde{\mathsf{Diff}}^+\mathbb{S}^1, \quad (\gamma, \theta) \mapsto \gamma \circ \theta$$

defines a group structure. Moreover, the group inversion

$$\iota : \widetilde{\mathsf{Diff}}^+\mathbb{S}^1 \to \widetilde{\mathsf{Diff}}^+\mathbb{S}^1, \quad \gamma \mapsto \iota(\gamma) = \gamma^{-1}$$

is the inversion of the functions and the neutral element is $\mathsf{id}_\mathbb{R}$.

Proof. Let $f, g \in \widetilde{\mathsf{Diff}}^+\mathbb{S}^1$. Then we have

$$(f \circ g)(t + 2\pi) = (f \circ g)(t) + 2\pi$$

and

$$|(f \circ g)'(t)| = |f'(g(t))| \cdot |g'(t)| > 0$$

for all $t \in \mathbb{R}$. This shows that the subset $\widetilde{\mathsf{Diff}}^+\mathbb{S}^1$ is closed under the composition map, which is associative. It remains to show, that every function $f \in \widetilde{\mathsf{Diff}}^+\mathbb{S}^1$ admits an inverse $f^{-1} \in \widetilde{\mathsf{Diff}}^+\mathbb{S}^1$. Since $f'(t) > 0$ the function $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing and therefore injective. Every value between $f(0)$ and $f(2\pi)$ is contained in the image of f, because f is continuous. Taking $f(t+2\pi) = f(t)+2\pi$ into account, we see that f is surjective, and hence bijective. This guarantees the existence of an inverse f^{-1}. By the Inverse Mapping Theorem and the fact that $f'(t) > 0$, it follows that the inverse f^{-1} is also smooth and fulfills $(f^{-1})'(t) > 0$. We conclude $f^{-1} \in \widetilde{\mathsf{Diff}}^+\mathbb{S}^1$. □

Definition 2.1.5 (Group of diffeomorphisms of the circle)**.**
Let $N \leq \widetilde{\mathsf{Diff}}^+\mathbb{S}^1$ be the central subgroup which consists of functions of the form

$$f : \mathbb{R} \to \mathbb{R}, \quad t \mapsto t + 2\pi \cdot k$$

with $k \in \mathbb{Z}$. We define the factor group

$$\mathsf{Diff}^+\mathbb{S}^1 := \widetilde{\mathsf{Diff}}^+\mathbb{S}^1/N.$$

Remark: In this sense the elements of $\mathsf{Diff}^+\mathbb{S}^1$ are equivalence classes of real-valued functions $f : \mathbb{R} \to \mathbb{R}$ with respect to the following equivalence relation. We say $f : \mathbb{R} \to \mathbb{R}$ is equivalent to $g : \mathbb{R} \to \mathbb{R}$ if there exists an integer $n \in \mathbb{Z}$ such that the difference

$$f - g \equiv n \cdot 2\pi.$$

For our proofs, it is convenient to view a real-valued function $f : \mathbb{R} \to \mathbb{R}$ as a representative of a diffeomorphism $[f] \in \mathsf{Diff}^+\mathbb{S}^1$. Now let us say some words about the derivative of a diffeomorphism. If $f \in \widetilde{\mathsf{Diff}}^+\mathbb{S}^1$, then by 2.1.3(i) the derivative $f' : \mathbb{R} \to \mathbb{R}$ is 2π-periodic, i.e, $f' \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$. The same is true for any diffeomorphism $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$; namely γ' is an element of $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$, because the difference between two the representatives of γ is an integer multiple of 2π. In this manner, the derivative of $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$ is well-defined.

Definition 2.1.6. We define $\mathsf{Rot}^+\mathbb{S}^1 \leq \mathsf{Diff}^+\mathbb{S}^1$ to be the subgroup of all diffeomorphisms such that every representative $f : \mathbb{R} \to \mathbb{R}$ of an element of $\mathsf{Rot}^+\mathbb{S}^1$ has the form

$$f(t) = t + C$$

where $C \in \mathbb{R}$ is a real number.

Remark: Let us say something about the topology of these groups. On $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$, there exists a natural Fréchet topology. If $f \in \widetilde{\mathsf{Diff}}^+\mathbb{S}^1$, then $f - \mathsf{id}_\mathbb{R}$ is an element of $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$. Therefore the single chart atlas

$$\widetilde{\mathsf{Diff}}^+\mathbb{S}^1 \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}), \quad f \mapsto f - \mathsf{id}_{\mathbb{S}^1}$$

defines a smooth structure on $\widetilde{\mathsf{Diff}}^+\mathbb{S}^1$. Since N is a discrete normal subgroup, $\mathsf{Diff}^+\mathbb{S}^1$ inherits its topology from its universal covering $\widetilde{\mathsf{Diff}}^+\mathbb{S}^1$. The group $\mathsf{Diff}^+\mathbb{S}^1$ is a Fréchet–Lie group [13] and so is $\widetilde{\mathsf{Diff}}^+\mathbb{S}^1$. In a similar manner, the homogeneous space $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ becomes a Fréchet manifold. The main result of this thesis, is that $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ has an invariant complex structure unique up to sign. The complex structure requires an atlas. The following sections, are devoted to introduce the subset $\mathsf{V}^+ \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$, which provides this.

2.2 Elementary Fourier Decomposition

This section provides us with some properties and definitions concerning the Fourier decomposition of 2π-periodic functions.

Lemma 2.2.1. *Let $f \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ be a 2π-periodic smooth function*

$$f : \mathbb{R} \to \mathbb{C}$$

with values in the complex numbers. Then f can be written as a Fourier series

$$f(t) := \sum_{n \in \mathbb{Z}} a_n e^{int}$$

with the Fourier coefficients

$$a_n = \frac{1}{2\pi} \int_{\mathbb{S}^1} e^{-int} f(t) dt.$$

This series converges uniformly.

Lemma 2.2.2. *Let $f : \mathbb{S}^1 \to \mathbb{C}$ be a continuous function and*

$$a_n = \frac{1}{2\pi} \int_{\mathbb{S}^1} e^{-int} f(t) dt$$

its Fourier coefficients, then

$$\|f\|_{L^2}^2 = \sum_{n \in \mathbb{Z}} |a_n|^2 < \infty.$$

Proof. This is a consequence of Theorem 1 on page 194 in [11]. $\square$

Definition 2.2.3. (i) We define $\mathtt{F}^+ \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ to be the subset of all smooth function $f : \mathbb{S}^1 \to \mathbb{C}$ of the form

$$f(t) = a_0 + a_1 e^{it} + a_2 e^{2it} + \cdots$$

with $a_0, a_1, a_2, \ldots \in \mathbb{C}$.

(ii) Similarly, let $\mathtt{F}^- \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ be the subset of functions of the form

$$f(t) = b_0 + b_1 e^{-it} + b_2 e^{-2it} + \cdots$$

with $b_0, b_1, b_2, \ldots \in \mathbb{C}$.

Lemma 2.2.4. *Let $f : \mathbb{S}^1 \to \mathbb{C}$ be a smooth map and*

$$f(t) = \sum_{n\in\mathbb{Z}} a_n e^{int}$$

its Fourier decomposition. Then

(i) *the Fourier decomposition of the derivative is*

$$f'(t) = \sum_{n\in\mathbb{Z}} (in) a_n e^{int},$$

(ii) $\sum_{n\in\mathbb{Z}} |a_n| < \infty$,

(iii) $\sum_{n\in\mathbb{Z}} |n|^k |a_n| < \infty$ *for all* $k \in \mathbb{N}_0$, *and*

(iv) *there exists a real constant* $C > 0$ *such that*

$$\|f\|_\infty \leq \|f\|_{L^2} + C\|f'\|_{L^2},$$

where $\|\cdot\|_\infty$ *is the sup-norm and* $\|\cdot\|_{L^2}$ *the* L^2*-norm.*

Proof. (i):

$$\int_0^{2\pi} e^{-int} f'(t) dt = e^{-int} f(t)\Big|_0^{2\pi} - \int_0^{2\pi} (-in)\, e^{-int} f(t) dt = (in) a_n.$$

(ii): Applying Lemma 2.2.2 to the Fourier series of f' to Lemma 2.2.2 results in

$$\sum_{n\in\mathbb{Z}} |n \cdot a_n|^2 < \infty. \qquad (\dagger)$$

We estimate

$$\sum_{n\in\mathbb{Z}} |a_n| = |a_0| + \sum_{0\neq n\in\mathbb{Z}} |\frac{1}{n}||n \cdot a_n| \leq |a_0| + \sqrt{\sum_{0\neq n\in\mathbb{Z}} \frac{1}{|n|^2}} \sqrt{\sum_{0\neq n\in\mathbb{Z}} |n a_n|^2} \overset{(\dagger)}{\leq} \infty$$

with the Cauchy-Schwartz inequality.

(iii): Just apply (i) and (ii) to the k-th derivative of f.

(iv):

$$\begin{aligned}
\|f\|_\infty &= \mathsf{sup}_{t\in\mathbb{S}^1} |\sum_{n\in\mathbb{Z}} a_n e^{int}| \\
&\leq \sum_{n\in\mathbb{Z}} |a_n| \\
&\leq |a_0| + \underbrace{\sqrt{\sum_{0\neq n\in\mathbb{Z}} \frac{1}{|n|^2}}}_{C} \sqrt{\sum_{=\neq n\in\mathbb{Z}} |na_n|^2} \\
&\leq \sqrt{\sum_{n\in\mathbb{Z}} |a_n|^2} + C \sqrt{\sum_{n\in\mathbb{Z}} |na_n|^2} \\
&= \|f\|_{L^2} + C\|f'\|_{L^2}.
\end{aligned}$$

□

2.3 Holomorphic Functions on the Unit Disc

In this section, we discuss basic properties of holomorphic functions particularly on the unit disk and their extensions to the boundary. Moreover, a function $f : \mathbb{S}^1 \to \mathbb{C}$ of the form $f(t) = a_0 + a_1 e^{it} + a_2 e^{2it} + \cdots$ can be considered as a holomorphic function by identifying the Taylor coefficients with the Fourier coefficients. This point of view will be elaborated in this section.

Definition 2.3.1 (Open unit disk)**.** We define

$$\mathbb{D} := \{\, z \in \mathbb{C} \,:\, |z| < 1 \,\}$$

to be the *open unit disk* of the complex numbers. We denote by

$$\overline{\mathbb{D}} := \{\, z \in \mathbb{C} \,:\, |z| \leq 1 \,\}$$

the *closed unit disk*, and by

$$\partial\mathbb{D} := \{\, z \in \mathbb{C} \,:\, |z| = 1 \,\}$$

its boundary.

Definition 2.3.2 (Complex conjugation)**.** We denote the *complex conjugation* of $z = x + iy \in \mathbb{C}$ by $\overline{z} = x - iy$.

Definition 2.3.3 (Domain)**.** We call a subset $G \subseteq \mathbb{C}$ of the complex plane a *domain* if it is open and connected.

Definition 2.3.4 (Biholomorphic)**.** A function $f : G_1 \to G_2$ between two domains G_1 and G_2 is called *biholomorphic* if f is holomorphic and f^{-1} is holomorphic as well.

In the case of one complex dimension, a biholomorphic map is sometimes called a *conformal isomorphism*.

Definition 2.3.5 (Univalent)**.** Let $G \subseteq \mathbb{C}$ be an open subset of the complex plane. A holomorphic function $f : G \to \mathbb{C}$ is called *univalent* if it is injective and $f'(z) \neq 0$ for all $z \in G$.

Remark: An equivalent characterization for univalent maps is that the corestriction $f : G \to f(G)$ to the image is biholomorphic.

Definition 2.3.6 (Smooth)**.** A continuous function $F : \overline{\mathbb{D}} \to \mathbb{C}$ is *continuously differentiable*, if there exists a continuous function $H : \overline{\mathbb{D}} \times \overline{\mathbb{D}} \to \mathbb{C}$ such that

$$F(z) - F(y) = (z - y) \cdot H(z, y)$$

for all $z, y \in \overline{\mathbb{D}}$. We call the function H the *difference quotient* and the function $F' : \overline{\mathbb{D}} \to \mathbb{C}$ with $F'(z) = H(z, z)$ the *derivative* of F. We call a function $F : \overline{\mathbb{D}} \to \mathbb{C}$ *twice continuously differentiable* if its derivative F' is continuously differentiable. In this way, we define a function to be *n-times continuously differentiable*. We call a function *smooth* if it is n-times continuously differentiable for all $n \in \mathbb{N}$.

Lemma 2.3.7. *Consider a continuous function $F : \overline{\mathbb{D}} \to \mathbb{C}$ whose restriction to the interior $\mathbb{D}$ is holomorphic. Then the following three statements are equivalent:*

(i) *The function $F : \overline{\mathbb{D}} \to \mathbb{C}$ is smooth in the sense of Definition 2.3.6.*

(ii) *For all $k \in \mathbb{N}$, the k-th derivative $F^{(k)} : \mathbb{D} \to \mathbb{C}$ considered as a holomorphic function has a continuous extension to the closure $\overline{\mathbb{D}}$.*

(iii) *The restriction of F to the boundary $\partial\mathbb{D}$ is smooth. This means that the function $f : \mathbb{S}^1 \to \mathbb{C}, \quad t \mapsto F(e^{it})$ is an element of $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$.*

Proof. (i)$\Longrightarrow$(ii): The statement (i) is stronger then (ii).
(ii)$\Longrightarrow$(i): Since $F' : \mathbb{D} \to \mathbb{C}$ has a continuous extension, F' is bounded on $\overline{\mathbb{D}}$. Therefore, the function

$$H : \overline{\mathbb{D}} \times \overline{\mathbb{D}} \to \mathbb{C}, \quad (z, y) \mapsto H(z, y) = \int_0^1 F'(y + t(z - y))dt$$

is continuous, and

$$\begin{aligned} F(z) - F(y) =& (z-y) \cdot \left[\frac{1}{(z-y)} F(y + t(z-y)) \right]_{t=0}^{t=1} \\ =& (z-y) \cdot \int_0^1 F'(y + t(z-y)) dt \\ =& (z-y) \cdot H(z,y) \end{aligned}$$

shows that it is the difference quotient. Hence, F fulfills the requirement of Definition 2.3.6 that F is continuously differentiable. We proceed inductively, and conclude that F is smooth in the sense of Definition 2.3.6.
(i)$\Longrightarrow$(iii): The restriction of a smooth function remains smooth.
(iii)$\Longrightarrow$(i): The Taylor coefficients of F coincide with the Fourier coefficients of f, because

$$\begin{aligned} \frac{1}{n!} F^{(n)}(0) &= \frac{1}{2\pi i} \oint_{\partial \mathbb{D}} \frac{F(z)}{z^{n+1}} dz = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{F(e^{it})}{e^{i(n+1)t}} i e^{it}\, dt \\ &= \frac{1}{2\pi} \int_{\mathbb{S}^1} f(t) e^{-int}\, dt. \end{aligned}$$

So, we can apply Lemma 2.3.8 and get that $F : \overline{\mathbb{D}} \to \mathbb{C}$ is smooth. □

Lemma 2.3.8. *Let $f : \mathbb{S}^1 \to \mathbb{C}$ be a smooth function and*

$$f(t) = a_0 + a_1 e^{it} + a_2 e^{2it} + \cdots$$

its Fourier series, i.e., $f \in \mathtt{F}^+$. Then the Taylor series

$$F(z) = a_0 + a_1 z + a_2 z^2 + \cdots .$$

defines a function $F : \overline{\mathbb{D}} \to \mathbb{C}$, which is smooth and the restriction to $\mathbb{D}$ is holomorphic.

Proof. 1. Step: $F : \overline{\mathbb{D}} \to \mathbb{C}$ is continuous and $F|_{\mathbb{D}}$ holomorphic:
Let $f \in \mathtt{F}^+$ and $f(t) = a_0 + a_1 e^{it} + a_2 e^{2it} + \cdots$ its Fourier expansion. Due to the inequality

$$\sum_{j \in \mathbb{Z}} |a_j| < \infty$$

of Lemma 2.2.4, the series

$$F(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

converges absolutely for $z \in \overline{\mathbb{D}}$. The function $F : \overline{\mathbb{D}} \to \mathbb{C}$ is continuous, because the partial sums of the Taylor series converge uniformly. Hence, the restriction to $\mathbb{D}$ is holomorphic and $F(e^{it}) = f(t)$ holds for $t \in \mathbb{R}$.
2. Step: $F : \overline{\mathbb{D}} \to \mathbb{C}$ is continuously differentiable:
Since f is smooth, the function

$$f_1(t) := -if'(e^{-it}) = a_1 + 2a_2 z + 3a_3 z^2 + \cdots$$

is also smooth. Applying the first Step to f_1, we see that

$$F'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \cdots$$

and therefore $F' : \overline{\mathbb{D}} \to \mathbb{C}$ is continuous. By induction, we obtain that for each $n \in \mathbb{N}$ the n-th derivative $F^{(n)} : \overline{\mathbb{D}} \to \mathbb{C}$ is continuous. This is statement (ii) of Lemma 2.3.7. Hence, $F : \overline{\mathbb{D}} \to \mathbb{C}$ is smooth in the sense of Definition 2.3.6. □

Lemma 2.3.9. *Let $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$. If the Fourier series of λ is*

$$\lambda(t) = b_1 e^{it} + b_2 e^{2it} + \cdots$$

with $b_1, b_2, \ldots \in \mathbb{C}$, then the map

$$\mathbb{S}^1 \to \mathbb{C}, \quad t \mapsto e^{\lambda(t)}$$

has a Fourier series of the form

$$e^{\lambda(t)} = 1 + a_1 e^{it} + a_2 e^{2it} + \cdots$$

with $a_1, a_2, \ldots \in \mathbb{C}$.

Proof. By Lemma 2.3.8, the function

$$L(z) := b_1 z + b_2 z^2 + \cdots$$

is holomorphic on $\mathbb{D}$ with $L(0) = 0$. The function $F(z) = e^{L(z)}$ is also holomorphic on $\mathbb{D}$ with $F(0) = 1$. Therefore, its Taylor series is

$$F(z) = 1 + a_1 z + a_2 z^2 + \cdots,$$

and hence

$$e^{\lambda(t)} = F(e^{it}) = a_0 + a_1 e^{it} + a_2 e^{2it} + \cdots.$$

□

2.4 Winding Number

In this section, we introduce the notion of winding number so that we can define the subsets $\mathtt{V}$, $\mathtt{V}^E$, $\mathtt{V}^+$, and $\mathtt{V}^-$ of the vector space $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$, which play a key role in this thesis.

Definition 2.4.1 (Winding number)**.** Let $f : \mathbb{S}^1 \to \mathbb{C}^\times$ be a piecewise continuously differentiable function into the punctured complex plane. We define the *winding number* of f by

$$\mathtt{w}(f) := \frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{f'(t)}{f(t)} dt.$$

Lemma 2.4.2. *Here are some properties of the winding number:*

(i) $\mathtt{w}(f \cdot g) = \mathtt{w}(f) + \mathtt{w}(g)$ *for* $f, g \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times)$.

(ii) $\mathtt{w}(\frac{1}{f}) = -\mathtt{w}(f)$.

(iii) *The winding number is an integer, i.e.,* $\mathtt{w}(f) \in \mathbb{Z}$.

(iv) *Invariance under re-parameterization, i.e.,*

$$\mathtt{w}(f) = \mathtt{w}(f \circ \gamma) \text{ for all } \gamma \in \mathsf{Diff}^+\mathbb{S}^1.$$

(v) *The function*

$$\mathtt{w} : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times) \to \mathbb{Z}, \quad f \mapsto \mathtt{w}(f)$$

is a group homomorphism, between $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times)$ *with the point-wise multiplication and* $(\mathbb{Z}, +)$.

Proof. (i):

$$\begin{aligned}
\mathtt{w}(f \cdot g) &= \frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{g(t) \cdot f'(t) + f(t) \cdot g'(t)}{f(t) \cdot g(t)} dt \\
&= \frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{f'(t)}{f(t)} dt + \frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{g'(t)}{g(t)} dt \\
&= \mathtt{w}(f) + \mathtt{w}(g).
\end{aligned}$$

(ii):

$$0 = w(1) = w(f \cdot \frac{1}{f}) = w(f) + w(\frac{1}{f}).$$

(iii): The definition of the winding number is an integral of the holomorphic function $\frac{1}{z}$ along a closed path

$$\mathtt{w}(f) = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{f'(t)}{f(t)} dt = \frac{1}{2\pi i} \oint_C \frac{1}{z} \, dz,$$

with $C = f(\mathbb{S}^1)$. Since the residue of $\frac{1}{z}$ at zero is $2\pi i$, the integral along an arbitrary path is a multiple of $2\pi i$.
(iv):

$$\mathtt{w}(f \circ \gamma) = \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)dt = \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{f'(u)}{f(u)}du = \mathtt{w}(f).$$

(v): This follows from (i)-(iii). □

Definition 2.4.3. We define $\mathtt{V} \subseteq \mathcal{C}^\infty(\mathbb{S}^1)_\mathbb{C}$ to be the subset of functions $f : \mathbb{S}^1 \to \mathbb{C}$ with the following properties:
(i) f is injective.
(ii) $f(t) \neq 0$ for all $t \in \mathbb{S}^1$.
(iii) $f'(t) \neq 0$ for all $t \in \mathbb{S}^1$.
(iv) $f : \mathbb{S}^1 \to \mathbb{C}^\times$ has winding number $\mathtt{w}(f) = 1$.
Moreover, we define $\mathtt{V}^E \subseteq \mathtt{V}$ to be the subset of smooth 2π-periodic functions $f : \mathbb{S}^1 \to \mathbb{C}^\times$ of the form

$$f(t) = e^{it}(r + a_1e^{it} + \cdots)$$

with Fourier coefficients $r \in \mathbb{R}^+$ and $a_1, a_2, \ldots \in \mathbb{C}$, and $\mathtt{V}^+ \subseteq \mathtt{V}$ to be the subset of functions $f : \mathbb{S}^1 \to \mathbb{C}^\times$ with Fourier series of the form

$$f(t) = e^{it}(1 + a_1e^{it} + a_2e^{2it} + \cdots),$$

where $a_1, a_2, \ldots$ are complex numbers, and $\mathtt{V}^- \subseteq \mathtt{V}$ to be the subset of function $g : \mathbb{S}^1 \to \mathbb{C}^\times$ with Fourier series of the form

$$g(t) = e^{it}(1 + b_1e^{-it} + b_2e^{-2it} + \cdots),$$

with $b_1, b_2, \ldots \in \mathbb{C}$.

Remark: Lemma 2.4.10 provides another characterization of the subset $\mathtt{V}^-$.

Definition 2.4.4. For a function $f \in \mathtt{V}$ we define the subset

$$\mathtt{D}_f := \{\, z \in \mathbb{C}\backslash f(\mathbb{S}^1) \;:\; \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{f'(t)}{f(t) - z}dt = 1 \,\},$$

and call it the *interior* of the curve f. We denote its boundary by $\partial\mathtt{D}_f$ and its closure by $\overline{\mathtt{D}}_f$.

Lemma 2.4.5 (Jordan Curve Theorem)**.** *For $f \in \mathtt{V}$*

(i) *the subset $\mathtt{D}_f$ is a simply connected domain of the complex plane,*

(ii) *its boundary $\partial\mathtt{D}_f$ is the image of the function $f : \mathbb{S}^1 \to \mathbb{C}$, and*

(iii) *the subset $\mathbb{C}\backslash\overline{\mathtt{D}}_f$ is a domain, whose boundary is $f(\mathbb{S}^1)$.*

Moreover,

$$\mathbb{C}\backslash\overline{\mathtt{D}}_f = \{\, z \in \mathbb{C}\backslash f(\mathbb{S}^1) \,:\, \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{f'(t)}{f(t)-z}dt = 0 \,\}.$$

Proof. By the Jordan Curve Theorem [21], the subset $\mathbb{C}\backslash f(\mathbb{S}^1)$ is the disjoint union of a bounded connected subset $A \subseteq \mathbb{C}\backslash f(\mathbb{S}^1)$ and a connected unbounded subset $B \subseteq \mathbb{C}\backslash f(\mathbb{S}^1)$. Since the function

$$\delta : \mathbb{C}\backslash f(\mathbb{S}^1) \to \mathbb{Z}, \quad z \mapsto \mathtt{w}(f-z) = \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{f'(t)}{f(t)-z}dt$$

is continuous, the two restrictions $\delta|_A$ and $\delta|_B$ are constant maps. If $z \in B$ is sufficiently large, the integrand

$$\frac{f'(t)}{f(t)-z}$$

becomes sufficiently small and we see that $\delta(z) = 0$. Furthermore, $\delta|_B \equiv 0$. Since $\mathtt{w}(f) = 1$ we have $0 \in \mathtt{D}_f$, which implies that $\mathtt{D}_f$ is not empty. Since $\delta_{\mathtt{D}_f} \equiv 1$, we have $\mathtt{D}_f \cap (B \cup f(\mathbb{S}^1)) = \emptyset$, and therefore $\mathtt{D}_f \subseteq A$. The function δ takes only two values. We conclude $\mathtt{D}_f = A$, and hence $\mathbb{C}\backslash\mathtt{D}_f = B$. □

Lemma 2.4.6. *Let $f \in \mathtt{V}^E$ with $f : \mathbb{S}^1 \to \mathbb{C}$ and let*

$$f(t) = e^{it}(r + a_1 e^{it} + a_2 e^{i2t} + \cdots)$$

its Fourier series, with $r \in \mathbb{R}^+$ and $a_1, a_2, a_3 \ldots \in \mathbb{C}$. Then the function $F : \mathbb{D} \to \mathtt{D}_f$ defined by the Taylor series

$$F(z) = z(r + a_1 z + a_2 z^2 + \cdots)$$

is biholomorphic.

Proof. By Lemma 2.3.8, the function $F : \mathbb{D} \to \mathbb{C}$ is holomorphic, and extends smoothly to the boundary. Let $H : \mathbb{D} \to \mathbb{C}$ be a holomorphic function which extends continuously to $\overline{\mathbb{D}}$. Then the Cauchy Theorem yields that

$$\frac{1}{2\pi i}\int_C \frac{H'(z)}{H(z)}dz = N$$

where N is the numbers of zero-values, lying in $\mathbb{D}$ counted with their multiplicity. Now we transfer this from zero-values to z_0-values. This means that

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{F'(z)}{F(z)-z_0}dz = \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{f'(t)}{f(t)-z_0}dt$$

counts how often the function $F : \mathbb{D} \to \mathbb{C}$ takes the values z_0. By Lemma 2.4.5, we have

$$\frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{f(t)}{f(t)-z_0}dt = \begin{cases} 1 & z_0 \in \mathtt{D}_f \\ 0 & z_0 \in \mathbb{C}\backslash\overline{\mathtt{D}}_f \end{cases} .$$

This shows that $F : \mathbb{D} \to \mathbb{C}$ is injective and its image is $\mathtt{D}_f = F(\mathbb{D})$, such that $F : \mathbb{D} \to \mathtt{D}_f$ is bijective. By Lemma 2.4.5, the subset $\mathtt{D}_f$ is open, which implies that F is not the constant function. We conclude $F : \mathbb{D} \to \mathtt{D}_f$ is biholomorphic. □

Lemma 2.4.7. *Let $F : \overline{\mathbb{D}} \to \mathbb{C}$ be a continuous function which is injective. Define $f(t) := F(e^{it})$. If*

(i) $F|_{\mathbb{D}}$ *is univalent,*
(ii) $F'(z) \neq 0$ *for all* $z \in \overline{\mathbb{D}}$*,*
(iii) $F|_{\partial\mathbb{D}}$ *is smooth,*
(iv) $F(0) = 0$*, and* $F'(0) = 1$*,*

then

$$f \in \mathtt{V}^+.$$

Proof. To show that $f \in \mathtt{V}$ we have to check the properties (i)-(iv) of Definition 2.4.3. The function $f : \mathbb{S}^1 \to \mathbb{C}$ is injective, because $F : \overline{\mathbb{D}} \to \mathbb{C}$ is injective. Since $F : \overline{\mathbb{D}} \to \mathbb{C}$ is injective and $F(0) = 0$, we obtain $f(t) \neq 0$ for all $t \in \mathbb{R}$. We have $f'(t) \neq 0$ for all $t \in \mathbb{R}$, because $F'(z) \neq 0$ holds for all $z \in \overline{\mathbb{D}}$. Next, we must compute the winding number of f. We have

$$\begin{aligned} \mathtt{w}(f) &:= \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{f'(t)}{f(t)}dt = \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{F'(e^{it})}{F(e^{it})}ie^{it}dt = \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{F'(z)}{F(z)}\,dz \\ &= \frac{1}{2\pi i}\oint_C \frac{1}{\xi}d\xi = 1 \end{aligned}$$

where C surrounds the set $F(\mathbb{D})$, which contains 0, because $F(0) = 0$ and F is injective. So far, it is shown that $f \in \mathtt{V}$. The restriction $F|_{\mathbb{D}}$ is univalent, $F(0) = 0$, and $F(0) = 1$, and therefore the Taylor series of F has the form

$$F(z) = z(1 + a_1 z + a_2 z^2 + \cdots).$$

Since $F : \overline{\mathbb{D}} \to \mathbb{C}$ is continuous the Fourier series of f reads

$$f(t) = F(e^{it}) = e^{it}(1 + a_1 e^{it} + a_2 e^{2it} + \cdots),$$

and $f \in \mathtt{V}^+$ is shown. □

Lemma 2.4.8. *If $f \in \mathtt{V}$ and $g(t) := 1/f(-t)$ for all $t \in \mathbb{S}^1$, then $g \in \mathtt{V}$.*

Proof. Recall the subset $\mathtt{V} \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ defined in 2.4.3. The function $g : \mathbb{S}^1 \to \mathbb{C}$ is injective, because f is injective. The function

$$g(t) = \frac{1}{f(-t)}$$

and its derivative

$$g'(t) = \frac{f'(-t)}{f(-t)^2}$$

are never zero. Finally, we compute the winding number

$$\mathtt{w}(g) = \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{g'(t)}{g(t)} dt = \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{f'(-t)}{f(-t)} dt = \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{f'(t)}{f(t)} dt = \mathtt{w}(f) = 1.$$

□

Definition 2.4.9. Now we can define the involutive function

$$\hat{\Upsilon} : V \to V, \quad f \mapsto \hat{\Upsilon}(f)$$

by

$$(\hat{\Upsilon}(f))(t) := \frac{1}{f(-t)} \text{ for } t \in \mathbb{S}^1.$$

Lemma 2.4.10. *The restriction*

$$\hat{\Upsilon} : \mathtt{V}^- \to \mathtt{V}^+, \quad g \mapsto \hat{\Upsilon}(g)$$

is bijective.

Proof. 1. Step: $f \in \mathtt{V}^+ \implies \hat{\Upsilon}(f) \in \mathtt{V}^-$:
Let $f(t) = e^{it}(1 + a_1 e^{it} + a_2 e^{2it} + \cdots)$, then by Lemma 2.4.6 the function

$$F(z) = z(1 + a_1 z + a_2 z^2 + \cdots)$$

is univalent on $\mathbb{D}$. Since $F(0) = 0$, we have

$$H(z) := \frac{F(z)}{z} = 1 + a_1 z + a_2 z^2 + \cdots \neq 0$$

for all $z \in \overline{\mathbb{D}}$. Therefore, the function $1/H(z)$ is holomorphic on $\mathbb{D}$ and its Taylor series is

$$\frac{1}{H(z)} = 1 + b_1 z + b_2 z^2 + \cdots .$$

This yields

$$\hat{\Upsilon}(f)(t) = \frac{1}{f(-t)} = \frac{1}{e^{-it} \cdot H(e^{-it})} = e^{it}(1 + b_1 e^{-it} + b_2 e^{-2it} + \cdots)$$

and we have shown $\hat{\Upsilon}(f) \in \mathtt{V}^-$.
2. Step: $g \in \mathtt{V}^- \Longrightarrow \hat{\Upsilon}(g) \in \mathtt{V}^+$:
Since $g(t) = e^{it}(1 + b_1 e^{-it} + b_2 e^{-2it} + \cdots)$, we have

$$h(t) := e^{it} \cdot g(-t) = 1 + b_1 e^{it} + b_2 e^{2it} + \cdots .$$

By Lemma 2.3.8, the function $H : \overline{\mathbb{D}} \to \mathbb{C}$ defined by

$$H(z) = 1 + b_1 z + b_2 z^2 + \cdots$$

is holomorphic on $\mathbb{D}$. The calculation

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{H'(z)}{H(z)} dz &= -\frac{1}{2\pi}\int_{\mathbb{S}^1} \frac{H'(e^{-it})}{H(e^{-it})} e^{-it} dt \\
&= -1 + \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{ie^{it} H(e^{-it}) - H'(e^{-it})}{e^{it} \cdot H(e^{-it})} dt \\
&= -1 + \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{g'(t)}{g(t)} dt = -1 + 1 \\
&= 0
\end{aligned}$$

implies that H has no zero-value on $\mathbb{D}$, and therefore

$$\frac{1}{H(z)} = 1 + a_1 z + a_2 z^2 + \cdots$$

is holomorphic. Hence,

$$\hat{\Upsilon}(g)(t) = \frac{1}{g(-t)} = \frac{e^{it}}{H(e^{it})} = e^{it}(1 + a_1 e^{it} + a_2 e^{2it} + \cdots),$$

and we conclude $\hat{\Upsilon}(g)(t) \in \mathtt{V}^+$. $\square$

Remark: Here we discuss some aspect concerning $\mathtt{V}^+$. The set $\mathtt{V}^+$ can be identified with the subset $S \subseteq \mathsf{Hol}(\mathbb{D}, \mathbb{C})$ of univalent functions $F : \mathbb{D} \to \mathbb{C}$ which have diffeomorphic extensions to the boundary and satisfy $F(0) = 0$ and $F'(0) = 1$. The Fourier series of such a function is

$$F(z) = z(1 + a_2 z^2 + a_3 z^3 + \cdots).$$

The *Koebe one-quarter theorem* [6] asserts that the range of F contains the disk

$$\{ z \in \mathbb{C} : |z| < \frac{1}{4} \}.$$

It must be said, that the *Koebe function*

$$K(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots$$

itself, which is the reason that $|F(-1)| = \frac{1}{4}$ is the best possible value, has an unbounded range due to the singularity at $z = 1$. Another aspect is the Bieberbach–de Branges Theorem [7], in former times known as the Bieberbach conjecture. It asserts that

$$|a_n| \leq n$$

holds for all $n > 1$. Equality holds for the Koebe function.

Chapter 3

Birkhoff Decomposition

Remember that $\mathtt{V}^+ \subseteq \mathtt{V}$ was defined in 2.4.3 as the subset of functions $f : \mathbb{S}^1 \to \mathbb{C}^\times$ with Fourier series of the form

$$f(t) = e^{it}(1 + a_1 e^{it} + a_2 e^{2it} + \cdots),$$

and $\mathtt{V}^- \subseteq \mathtt{V}$ the subset of functions $g : \mathbb{S}^1 \to \mathbb{C}^\times$ with

$$g(t) = e^{it}(1 + a_1 e^{-it} + a_2 e^{-2it} + \cdots),$$

where $a_1, a_2, \ldots$ are complex numbers. In this chapter, we prove that for every diffeomorphism $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$, there exist two functions $f \in \mathtt{V}^+$ and $g \in \mathtt{V}^-$ and a complex number c such that

$$f \circ \gamma = c \cdot g.$$

We call this the Birkhoff decomposition of γ. This name is motivated in Section 1 by its analogy to the the Birkhoff decomposition of the Lie-group $\mathsf{SU}(1,1)$, where $\mathsf{SU}(1,1)$ is acting by Möbius transformations on the Riemann sphere $\overline{\mathbb{C}}$. In Section 2, we analyze the regularity of the solution $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the Beltrami equation

$$f_{\bar{z}} = \mu \cdot f_z$$

in dependence on the regularity of the Beltrami coefficient $\mu : \overline{\mathbb{C}} \to \mathbb{C}$. To handle this question, we rewrite the Beltrami equation in the form of an elliptic partial differential equation. In Section 3, we construct for a given diffeomorphism $\tilde{\gamma} \in \mathsf{D\dot{i}ff}(\partial\mathbb{D})$ an N-times continuously differentiable extension $\Gamma_N : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ onto the interior of the unit disk. In Section 4, we derive the Birkhoff decomposition with the help of Section 2 and Section 3.

The proof of this Birkhoff decomposition for analytic[1] diffeomorphisms on $\partial\mathbb{D}$ was given by Kirillov [16]. Here, we give a sketch of his proof. Kirillov cuts the Riemann sphere on the equator into two hemi spheres – say the northern and southern hemi sphere – both with the equator as their boundaries. A point z of the equator of the northern hemi sphere is identified with the equator point $\tilde{\gamma}(z)$ of the southern hemi sphere. This leads to a new sphere. On the new sphere, the complex structure can be extended beyond the equator, because $\tilde{\gamma}$ can be extended to a holomorphic function on a neighborhood of $\partial\mathbb{D}$. In this way, the new sphere is isomorphic as a complex manifold to the Riemann sphere. The old equator on the new Riemann sphere is just an analytic Jordan curve. This Jordan curve divides the new Riemann sphere into two domains. These two domains determine (via the Riemann Mapping Theorem) the functions f and g. Kirillovs argument fails for an arbitrary diffeomorphism, because after gluing the hemi spheres together, the new sphere has no smooth structure without further ado.

3.1 Motivation

Consider the group $\mathsf{SU}(1,1)$. It is the group of matrices of the form

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$$

with $a, b \in \mathbb{C}$ obeying $a\overline{a} - b\overline{b} = 1$. Every element of $\mathsf{SU}(1,1)$ can be written as a product like

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\overline{b}}{a} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}$$

called the Birkhoff decomposition. It should be noted that only the middle factor lies in $\mathsf{SU}(1,1)$. The other two factors are triangular matrices, which are not elements of $\mathsf{SU}(1,1)$ except unit matrix, but elements of the complexification $\mathsf{SL}(2,\mathbb{C})$.

The relation between $\mathsf{SU}(1,1)$ and $\mathsf{Diff}^+\mathbb{S}^1$

Consider the realization of $\mathsf{GL}(2,\mathbb{C})$ by Möbius transformations

$$\mathsf{GL}(2,\mathbb{C}) \to \mathsf{Aut}(\overline{\mathbb{C}}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \{z \mapsto \frac{az+b}{cz+d}\}$$

[1] Even if the article (a translation from the Russian) talks about the diffeomorphism $\tilde{\gamma}$ the proof given there works only for an analytic $\tilde{\gamma}$.

on the Riemann sphere $\overline{\mathbb{C}}$. If we restrict the Möbius transformations to the subgroup $\mathsf{SU}(1,1)$, then we have 3 orbits on $\overline{\mathbb{C}}$. The circle $\partial\mathbb{D}$, the open unit disk $\mathbb{D}$, and the rest $\overline{\mathbb{C}}\backslash\overline{\mathbb{D}}$. We are interested in the circle $\partial\mathbb{D}$, which corresponds to the one-sphere via

$$\eta : \mathbb{S}^1 \to \partial\mathbb{D}, \quad t \mapsto e^{it}.$$

This consideration yields a group homomorphism

$$\phi : \mathsf{SU}(1,1) \to \mathsf{Diff}^+\mathbb{S}^1, \quad \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \mapsto \{t \mapsto \eta^{-1}\left(\frac{ae^{it}+b}{\overline{b}e^{it}+\overline{a}}\right)\}$$

with kernel $\mathbb{Z}_2 = \{\mathbb{1}, -\mathbb{1}\}$. Here, $\mathbb{1}$ denotes the unit matrix.

Transfer to $\mathsf{Diff}^+\mathbb{S}^1$

Since, in contrast to $\mathsf{SU}(1,1)$, the diffeomorphism group $\mathsf{Diff}^+\mathbb{S}^1$ admits no complexification [19, 26], we have to explain where the factors of the Birkhoff decomposition live. To motivate this, we give a geometric interpretation of the product

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\bar{b}}{a} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}$$

as a composition of Möbius transformations

$$\phi\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} = \phi\begin{pmatrix} 1 & 0 \\ \frac{\bar{b}}{a} & 1 \end{pmatrix} \circ \phi\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \circ \phi\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}.$$

The transformation

$$g := \phi\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}, \quad z \mapsto z\left(1 + \frac{b}{a}\cdot\frac{1}{z}\right)$$

has a fixed point at $\infty \in \overline{\mathbb{C}}$, because the restriction to $\mathbb{C}$ is an affine map. The unit circle $\partial\mathbb{D}$ is mapped to another curve K_γ on the Riemann sphere. The second transformation

$$\phi\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}, \quad z \mapsto a^2 z$$

is just a multiplication with a complex number $a^2 \in \mathbb{C}$ with $|a^2| \geq 1$. Regarding K_γ as a curve in the complex plane, it will be enlarged and rotated. The third transformation

$$\phi\begin{pmatrix} 1 & 0 \\ \frac{\bar{b}}{a} & 1 \end{pmatrix} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}, \quad z \mapsto z(1 + \frac{\bar{b}}{a}z)^{-1}$$

has a fixed point at $0 \in \overline{\mathbb{C}}$. It maps K_γ back onto the unit disk $\partial\mathbb{D}$. In summary the unit disk $\partial\mathbb{D}$ was mapped onto itself and a diffeomorphism on $\partial\mathbb{D}$ was created. The main result of this chapter is that any diffeomorphism of $\partial\mathbb{D}$ can be obtained in the way described above. We need 3 mappings. The first one g is holomorphic on $\overline{\mathbb{C}}\backslash\overline{\mathbb{D}}$, and its extension maps $\partial\mathbb{D}$ onto another curve K_γ. The behavior on $\mathbb{D}$ remains undefined. The second mapping is just a multiplication by a complex number $c \in \mathbb{C}$. The third is a holomorphic mapping[2] f^{-1}, which maps the interior of K_γ onto the open unit disk $\mathbb{D}$, and the curve K_γ onto its boundary $\partial\mathbb{D}$. In summary we have

$$\gamma = f^{-1} \circ c \cdot g.$$

3.2 Solving the Beltrami Equation

In this section, we will show that the solution $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the Beltrami equation

$$f_{\bar{z}} = \mu \cdot f_z$$

is a $\mathcal{C}^N$-isomorphism if μ is $(N+2)$-times continuously differentiable (Proposition 3.2.18). The strategy is to rewrite the Beltrami equation into a system of two elliptic partial differential equation (see the proof of Lemma 3.2.16), so that we can exploit the theory of elliptic partial differential equations.

3.2.1 Quasi-Conformal Mappings

The existence of a solution of the Beltrami equation is investigated in the theory of of quasi-conformal mappings. In this subsection, we recall fundamental concepts and some results about quasi-conformal mappings for our purpose.

Definition 3.2.1 (Test functions)**.** Consider an open subset $U \subseteq \mathbb{R}^2$. A smooth function $f : U \to \mathbb{R}$ with compact support is called a *test function*. We denote by $\mathcal{D}(U)$ the vector space of all test functions on U. We now construct the topology on $\mathcal{D}(U)$. For every compact subset $K \subseteq U$, let

$$\mathcal{D}_K(U) = \{\, f \in \mathcal{D}(U) \,:\, \mathsf{supp}\, f \subseteq K \,\}$$

be the vector subspace of the functions whose support is a subset of K. For every compact subset $K \subseteq U$ and every tuple of natural numbers (r, s) we

[2]We write f^{-1} instead of f to create the same setup as used later on.

define a semi-norm by

$$p_{K,r,s}(f) := \mathsf{sup}_{(x,y)\in K}\left|\frac{\partial^{r+s}}{\partial x^r \partial y^s} f(x,y)\right|.$$

The family $\{p_{K,r,s}\}_{r,s\in\mathbb{N}_0}$ of semi-norms defines a Fréchet topology on $\mathcal{D}_K(U)$. The topology on the test functions $\mathcal{D}(U)$ is the inductive limit topology with respect to the inclusions

$$\mathcal{D}_K(U) \to \mathcal{D}(U).$$

Definition 3.2.2 (Distribution)**.** Consider an open subset $U \subseteq \mathbb{R}^2$ and the vector space of test-functions $\mathcal{D}(U)$ endowed with the topology defined above. A distribution is defined to be a continuous linear map

$$u : \mathcal{D}(U) \to \mathbb{R}.$$

We denote the vector space of distributions by $\mathcal{D}'(U)$.

Definition 3.2.3 (Locally integrable)**.** Let $U \subseteq \mathbb{R}^2$ be an open subset. A measurable function $f : U \to \mathbb{R}$ is called *locally integrable* if for every compact subset $K \subseteq U$ the integral

$$\int_K |f(x,y)|\, dxdy$$

is defined and finite.

Remark: Every locally integrable function u as well as every continuous function u defines a distribution by

$$\int_U u(x,y)\cdot f(x,y)dxdy$$

for all $f \in \mathcal{D}(U)$

Definition 3.2.4 (Derivative of a distribution)**.** Let $u \in \mathcal{D}'(U)$ be a distribution on the open subset $U \subseteq \mathbb{R}^2$. Its partial derivative $u_x \in \mathcal{D}'(U)$ is defined by

$$u_x(f) := -u(f_x)$$

for all $f \in \mathcal{D}(U)$, and $u_y \in \mathcal{D}'(U)$ is defined in the same way.

Remark: In this way, for every continuous map the derivative is well-defined, and we call it the *derivative in the distributional sense*.

Definition 3.2.5 (Quasi-conformal)**.** Let $U \subseteq \mathbb{C}$ be an open subset. A homeomorphism $f : U \to f(U) \subseteq \mathbb{C}$ is called *quasi-conformal*, if there exists a real positive number $k < 1$ such that

(i) the distributional derivatives

$$f_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)f(x+iy) \qquad \text{and} \qquad f_z = \frac{1}{2}(\partial_x - i\partial_y)f(x+iy)$$

on U are locally integrable and

(ii) $|f_{\bar{z}}| < k \cdot |f_z|$ almost everywhere.

Remark: The concept of quasi-conformal maps is not a local concept, since the extensions to arbitrary Riemann surfaces requires that k is keept globally fixed.

Definition 3.2.6 (Beltrami coefficient)**.** Let $U_1, U_2 \subseteq \mathbb{C}$ be open subsets of the complex plane and $f : U_1 \to U_2$ be a quasi-conformal map. The almost everywhere defined[3] function $\mu : U_1 \to \mathbb{C}$ which fullfils

$$f_z(z) = \mu(z) f_{\bar{z}}(z)$$

almost everwhere is called the *complex dilatation* or *Beltrami coefficient* of f.

Definition 3.2.7 (Beltrami equation)**.** The partial differential equation

$$f_{\bar{z}}(z) = \mu(z) \cdot f_z(z)$$

is called *Beltrami equation* with *Beltrami coefficient* μ.

Lemma 3.2.8 (Mapping theorem)**.** *Let $\mu : \overline{\mathbb{C}} \to \mathbb{C}$ be a Lebesgue measurable function with $\|\mu\|_\infty < 1$, then there exists a quasi-conformal map*

$$f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$$

which is a solution of the Beltrami equation

$$f_{\bar{z}}(z) = \mu(z) \cdot f_z(z)$$

with the fixed points 0,1 *and* ∞*. Moreover, this solution of f is unique.*

Proof. [25] Theorem 1 on page 10. ∎

[3] We use the teminology of [25].

Lemma 3.2.9. *Let $U \subseteq \mathbb{C}$ be an open subset and $f : U \to \mathbb{C}$ a continuously differentiable function. Then the following statements are equivalent:*

(i) $|\partial_{\bar{z}} f| < |\partial_z f|$.

(ii) *The Jacobian determinant is positive, i.e.,*

$$\operatorname{det} df = \operatorname{det} \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} > 0$$

with $f = u + iv$.

Proof. Let $f = u + iv$, then

$$\begin{aligned} \partial_z f &= \frac{1}{2}(\partial_x - i\partial_y)(u + iv) = \frac{1}{2}(u_x + v_y) + \frac{i}{2}(v_x - u_y), \\ \partial_{\bar{z}} f &= \frac{1}{2}(\partial_x + i\partial_y)(u + iv) = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y). \end{aligned}$$

Therefore,

$$|\partial_{\bar{z}} f| < |\partial_z f|$$

is equivalent to

$$(u_x - v_y)^2 + (v_x + u_y)^2 < (u_x + v_y)^2 + (v_x - u_y)^2,$$

which is equivalent to

$$0 < u_x v_y - v_x u_y = \operatorname{det} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

□

Lemma 3.2.10. *Let $U \subseteq \mathbb{C}$ be an open subset. Assume $f : U \to \mathbb{C}$ and $g : U \to \mathbb{C}$ are two continuously differentiable solutions of the Beltrami equation*

$$f_{\bar{z}}(z) = \mu(z) \cdot f_z(z) \qquad \textit{respectively} \qquad g_{\bar{z}}(z) = \mu(z) \cdot g_z(z)$$

for a given complex dilatation $\mu : U \to \mathbb{C}$ with $\|\mu\|_\infty < 1$. If $g : U \to g(U)$ is bijective and its inverse g^{-1} is continuously differentiable, then

$$f \circ g^{-1} : g(U) \to f(U)$$

is a holomorphic map.

Proof. The complex chain rule yields

$$\begin{aligned}\partial_{\bar z}(f \circ g^{-1}) &= \partial_z f \cdot \partial_{\bar z} g^{-1} + \partial_{\bar z} f \cdot \partial_{\bar z} \overline{g^{-1}} \\ &= \partial_z f \cdot (\partial_{\bar z} g^{-1} + \mu\, \partial_{\bar z} \overline{g^{-1}}).\end{aligned}$$

Since $g \circ g^{-1} = \mathsf{id}$, we obtain

$$0 = \partial_{\bar z}(g \circ g^{-1}) = \partial_z g \cdot (\partial_{\bar z} g^{-1} + \mu\, \partial_{\bar z} \overline{g^{-1}})$$

in a similar way. Due to $\partial_z g = 0$, we conclude

$$\partial_{\bar z}(f \circ g^{-1}) = 0,$$

which means that $f \circ g^{-1}$ is holomorphic. □

Lemma 3.2.11. *Let $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a biholomorphic map on the Riemann sphere. If $\varphi(0) = 0$, $\varphi'(0) = 1$ and $\varphi(\infty) = \infty$, then φ is the identity map.*

Proof. All biholomorphic maps of the Riemann sphere are Möbius transformations of the form

$$\varphi(z) = \frac{az + b}{cz + 1}.$$

The condition $\varphi(\infty) = \infty$ implies $c = 0$, $\varphi(0) = 0$ implies $b = 0$ and $\varphi'(0) = 1$ implies $a = 1$. Hence, $\varphi(z) = z$. □

3.2.2 Regularity of the Beltrami Equation

Before we study the regularity of the solutions of the Beltrami equation, we recall some results (3.2.12-3.2.15) from the theory if partial differential equations.

Lemma 3.2.12. *Consider the Beltrami equation*

$$f_{\bar z}(z) = \mu(z) \cdot f_z(z)$$

for an open subset $U \subseteq \mathbb{C}$. Let $p > 2$. If $\mu_z \in L^p(U)$, then the quasi-conformal solution f is $\mathcal{C}^1$.

Proof. [1] Lemma 3 on page 94. □

Definition 3.2.13. Let m be a natural number and let $U \subseteq \mathbb{R}^n$ be an open subset. We denote by $H^m_{loc}(U)$ the set of functions with the property that for every point $x \in U$ there exists an open neighborhood $V_x \subseteq U$ such that

$$f|_{V_x} \in H^m(V_x).$$

Here H^m is the Sobolev space $W^{m,2}$, which is a Hilbert space.

Definition 3.2.14 (Elliptic operator)**.** Let $U \subseteq \mathbb{R}^n$ be an open subset and $x_1, \ldots, x_n$ coordinates. A differential operator on U of the form

$$L = \sum_{i,j=1}^{n} a_{ij}(x_1, \ldots, x_n) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} b_j(x_1, \ldots, x_n) \frac{\partial}{\partial x_j} + c(x_1, \ldots, x_n)$$

with $a_{ij} = a_{ji}$ is called an *elliptic operator*, if there exists a real number $\epsilon > 0$ such that for every point $x \in U$ all eigenvalues of the matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

are greater than ϵ.

Lemma 3.2.15. *Let $U \subseteq \mathbb{R}^n$ be an open subset and m a non-negative integer. Let*

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} + c\cdot$$

be an elliptic operator with $a_{ij}, b_i, c \in \mathcal{C}^{m+1}(U)$. Moreover, let $h \in H^m(U)$. If $u \in H^1(U)$ is a weak solution of the elliptic partial differential equation

$$La = \sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j u + \sum_{i=1}^{n} b_i \partial_i u + c \cdot u = h,$$

then

$$u \in H^{m+2}_{loc}(U).$$

Proof. [10] Theorem 2 on page 314. $\square$

Lemma 3.2.16. *Let $U \subseteq \mathbb{C}$ be an open subset, $m > 1$ a positive integer, and $\mu \in \mathcal{C}^{m+2}(U)$ with $\|\mu\|_\infty < 1$. If $f \in H^{m+1}(U)$ is a solution of the Beltrami equation*

$$f_{\bar{z}}(z) = \mu(z) \cdot f_z(z),$$

then

$$f \in H^{m+2}_{loc}(U).$$

Proof. Split f and μ in its real and imaginary part like

$$f(x+iy) = u(x,y) + iv(x,y) \quad \text{and} \quad \mu(x+iy) = \mu_1(x,y) + i\mu_2(x,y).$$

Define the two operators

$$\nabla_1 := (1-\mu_1)\partial_x - \mu_2\partial_y \quad \text{and} \quad \nabla_2 := -\mu_2\partial_x + (1+\mu_1)\partial_y.$$

Due to

$$\begin{aligned} \partial_{\bar{z}} - \mu\partial_z &= \frac{1}{2}(\partial_x + i\partial_y) - (\mu_1 + i\mu_2)\frac{1}{2}(\partial_x - i\partial_y) \\ &= \frac{1}{2}[(1-\mu_1)\partial_x - \mu_2\partial_y] + i[-\mu_2\partial_x + (1+\mu_1)\partial_y] \\ &= \frac{1}{2}(\nabla_1 + i\nabla_2) \end{aligned}$$

we can write the Beltrami equation as

$$(\nabla_1 + i\nabla_2)(u+iv) = 0. \qquad (\dagger)$$

Applying $\nabla_1 - i\nabla_2$ to equation ($\dagger$) results in

$$(\nabla_1 - i\nabla_2)(\nabla_1 + i\nabla_2)(u+iv) = 0.$$

Now we separate the real from the imaginary part, and obtain two equations

$$L.u = [\nabla_1, \nabla_2].v, \qquad L.v = -[\nabla_1, \nabla_2].u$$

with the operator $L := \nabla_1^2 + \nabla_2^2$. Now we will show that the operator

$$\begin{aligned} L &= \nabla_1^2 + \nabla_2^2 \\ &= [(1-\mu_1)\partial_x - \mu_2\partial_y][(1-\mu_1)\partial_x - \mu_2\partial_y] \\ &\quad + [-\mu_2\partial_x + (1+\mu_1)\partial_y][-\mu_2\partial_x + (1+\mu_1)\partial_y] \\ &= (\underbrace{(1-\mu_1)^2 + \mu_2^2}_{a_{11} \in \mathcal{C}^{m+2}(U)})\partial_x^2 + 2[\underbrace{(1-\mu_1)(-\mu_2) - \mu_2(1+\mu_1)}_{a_{12}=a_{21} \in \mathcal{C}^{m+2}(U)}]\partial_x\partial_y \\ &\qquad + (\underbrace{\mu_2^2 + (1+\mu_1)^2}_{a_{22} \in \mathcal{C}^{m+2}(U)})\partial_y^2 \\ &\quad + \underbrace{(-\nabla_1\mu_1 - \nabla_2\mu_2)}_{=:b_1 \in \mathcal{C}^{m+1}(U)}\partial_x + \underbrace{(-\nabla_1\mu_2 + \nabla_2\mu_1)}_{=:b_2 \in \mathcal{C}^{m+1}(U)}\partial_y, \end{aligned}$$

is elliptic. The coefficient matrix of L is

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} (1-\mu_1)^2+\mu_2^2 & -2\mu_2 \\ -2\mu_2 & (1+\mu_1)^2+\mu_2^2 \end{pmatrix},$$

and its characteristic polynomial is

$$\begin{aligned} \mathsf{det}(a_{ij}-\delta_{ij}\lambda) &= ((1-\mu_1)^2+\mu_2^2-\lambda)((1+\mu_1)^2+\mu_2^2-\lambda)-4\mu_2^2 \\ &= (1+|\mu|^2-\lambda)^2-4|\mu|^2. \end{aligned}$$

The eigenvalues, which are the roots of the characteristic polynomial, are obtained to be

$$\begin{aligned} \lambda_{1,2} &= 1+|\mu|^2 \pm 2|\mu| \\ &= (1\pm|\mu|)^2. \end{aligned}$$

They have a uniform lower bound, greater than zero, due to the assumption $|\mu| \le \|\mu\|_\infty < 1$. So far, we have shown that L is elliptic. Next, we will show that

$$h := [\nabla_1, \nabla_2].v \in H^m_{loc}(U).$$

The commutator is

$$\begin{aligned} [\nabla_1, \nabla_2] &= \underbrace{\left[-(1-\mu_1)(\partial_x\mu_2)+\mu_2(\partial_y\mu_2)+\mu_2(\partial_x\mu_1)-(1+\mu_1)(\partial_y\mu_1)\right]}_{=:\xi_1}\partial_x \\ &\quad + \underbrace{\left[-(1-\mu_1)(\partial_x\mu_1)-\mu_2(\partial_y\mu_1)-\mu_2(\partial_x\mu_2)+(1+\mu_1)(\partial_y\mu_2)\right]}_{=:\xi_2}\partial_y. \end{aligned}$$

The assumption $\mu_1, \mu_2 \in \mathcal{C}^{m+2}(U)$ implies $\xi_1, \xi_2 \in \mathcal{C}^{m+1}(U)$. Taking into account that $v \in H^{m+1}(U)$, we have shown that

$$[\nabla_1, \nabla_2].v \in H^m(U)$$

holds. Moreover, $u \in H^1(U)$. Now we can apply Lemma 3.2.15 to the equation

$$L.u = h$$

and obtain $u \in H^{m+2}_{loc}(U)$. The argument for $v \in H^{m+2}_{loc}(U)$ is similar, and we conclude

$$f \in H^{m+2}_{loc}(U).$$

□

Lemma 3.2.17 (Sobolev's Lemma)**.** *Suppose that N,n,m are integers, $n > 0$, $N \geq 0$ and*

$$m > N + \frac{n}{2}.$$

If $U \subseteq \mathbb{R}^n$ is an open subset, then the Sobolev space $H^m(U)$ is a subspace of $\mathcal{C}^N(U)$.

Proof. See [29] on page 185. $\square$

Proposition 3.2.18. *Let $N > 2$ be a natural number and $\mu : \overline{\mathbb{C}} \to \mathbb{C}$ an $(N+2)$-times continuously differentiable function with $\|\mu\|_\infty < 1$. Then there exists a $\mathcal{C}^N$-isomorphism*

$$\tilde{f} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$$

with $\tilde{f}(0) = 0$, $\tilde{f}'(0) = \tilde{f}_z(0) = 1$ and $\tilde{f}(\infty) = \infty$ which is a solution of the Beltrami equation

$$\tilde{f}_{\bar{z}} = \mu \cdot \tilde{f}_z.$$

A $\mathcal{C}^N$-isomorphism is an n-times continuouly differentiable bijective map whose inverse is also n-times continuouly differentiable.

Proof. 1. Step: existence of a $\mathcal{C}^1$-solution:
Since $\mu : \overline{\mathbb{C}} \to \mathbb{C}$ is N-times continuously differentiable and

$$\|\mu\|_\infty < 1,$$

the assumptions for the Mapping Theorem 3.2.8 are satisfied. This yields a quasi-conformal solution $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the Beltrami equation with

$$f(0) = 0, \quad f(1) = 1 \quad \text{and } f(\infty) = \infty,$$

For every $x \in \overline{\mathbb{C}}$ there exists an open neighborhood U such that

$$(\partial_z \mu)|_U \in L^p(U) \qquad \text{for} \qquad p > 2$$

and by Lemma 3.2.12 we have $f \in \mathcal{C}^1(U)$, which implies $f \in H^1_{loc}(U)$ and therefore $f \in H^1_{loc}(\overline{\mathbb{C}})$.
2. Step: $f \in H^{N+2}_{loc}(\overline{\mathbb{C}})$:
To prove this, we show

$$n \leq N+1 \Longrightarrow f \in \ H^{n+1}_{loc}(\overline{\mathbb{C}}) \qquad (\dagger)$$

by induction on n. Since $f \in H^1_{loc}(\overline{\mathbb{C}})$, the case $n = 0$ is done. Now we will establish the induction from n to $n+1$, where we need only to consider the case $n+1 \leq N+1$. For every $x \in \mathbb{C}$, there exists an open neighborhood $U \subseteq \overline{\mathbb{C}}$ such that

$$f|_U \in H^{n+1}(U),$$

because $f \in H^{n+1}_{loc}(\overline{\mathbb{C}})$. Since $\mu \in \mathcal{C}^{n+2}(U) \subseteq \mathcal{C}^{N+2}(U)$ we conclude

$$f|_U \in H^{n+2}_{loc}(U)$$

by Lemma 3.2.16. Hence, $f \in H^{n+2}_{loc}(\overline{\mathbb{C}})$ and the induction is complete.
3. Step:
Due to Sobolev's Lemma 3.2.17, the second Step yields $f \in \mathcal{C}^N(\overline{\mathbb{C}})$. Let u be the real and v the imaginary part of f. The assumption $\|\mu\|_\infty < 1$ implies

$$|\partial_{\bar{z}} f| < |\partial_z f|,$$

which is equivalent to

$$\mathsf{det}\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = u_x v_y - v_x u_y > 0$$

by Lemma 3.2.9. We conclude that f is a $\mathcal{C}^N$-isomorphism, and

$$\tilde{f}(z) := \frac{f(z)}{f'(0)}$$

is a $\mathcal{C}^N$-isomorphism, which fulfills the assertion of the proposition. $\square$

3.3 Constructing Auxiliary Functions

Let $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$ and $\tilde{\gamma} : \partial\mathbb{D} \to \partial\mathbb{D}$ the corresponding diffeomorphism with winding number 1 such that

$$\tilde{\gamma}(e^{it}) = e^{i\gamma(t)}$$

for all $t \in \mathbb{R}$. In this section, we construct a quasi-conformal function $\Gamma_N : \overline{\mathbb{D}} \to \mathbb{C}$ with $\Gamma_N|_{\partial\mathbb{D}} = \tilde{\gamma}$ such that all partial derivatives of the complex dilatation

$$\mu_N = \frac{\partial_{\bar{z}}\Gamma_N}{\partial_z \Gamma_N}$$

up to order N are zero on $\partial\mathbb{D}$. During this section, γ and the natural number $N > 2$ are fixed.

3.3.1 Bump-Functions

Definition 3.3.1 (Bump function)**.** We define the well-known *bump function* by

$$b : \mathbb{R} \to \mathbb{R}, \quad x \mapsto b(x) := \frac{b_+(\frac{3}{2} - 2x)}{b_+(\frac{3}{2} - 2x) + b_+(2x - \frac{1}{2})}$$

with

$$b_+(\xi) := \begin{cases} e^{-\frac{1}{\xi^2}} & \text{for} \quad \xi > 0 \\ 0 & \text{for} \quad \xi \leq 0. \end{cases}$$

Lemma 3.3.2. *The bump function $b : \mathbb{R} \to \mathbb{R}$ has the following properties:*

(i) *Symmetry:*

$$b(1 - x) = 1 - b(x).$$

(ii) *Values:*

$$\begin{aligned} b(x) = 1 \quad &\textit{for} \quad x \leq \frac{1}{4}, \\ b(x) = 0 \quad &\textit{for} \quad x \geq \frac{3}{4}. \end{aligned}$$

(iii)

$$\mathsf{min}_{x \in \mathbb{R}} b'(x) = b'(\frac{1}{2}) = -16.$$

Proof. (i):

$$\begin{aligned} b(1 - x) &= \frac{b_+(\frac{3}{2} - 2(1 - x))}{b_+(\frac{3}{2} - 2(1 - x)) + b_+(2(1 - x) - \frac{1}{2})} \\ &= \frac{b_+(2x - \frac{1}{2})}{b_+(2x - \frac{1}{2}) + b_+(\frac{3}{2} - 2x)} \\ &= 1 - \frac{b_+(\frac{3}{2} - 2x)}{b_+(2x - \frac{1}{2}) + b_+(\frac{3}{2} - 2x)} \\ &= 1 - b(x). \end{aligned}$$

(ii):

If $x \leq \frac{1}{4}$, then $b_+(\frac{3}{2} - 2x) = 0$, and therefore $b(x) = 0$. The case $x > \frac{3}{4}$ follows from (i).
(iii):

$$\begin{aligned}
b'_+(\xi) &= \frac{2}{\xi^3} e^{-\frac{1}{\xi^2}} = \frac{2}{\xi^3} b_+(\xi), \\
b'(x) &= -2 \frac{b'_+(\frac{3}{2} - 2x) b_+(2x - \frac{1}{2}) + b_+(\frac{3}{2} - 2x) b'_+(2x - \frac{1}{2})}{[b_+(\frac{3}{2} - 2x) + b'_+(2x - \frac{1}{2})]^2}, \\
b'(\frac{1}{2}) &= -2 \frac{b'_+(\frac{1}{2}) b_+(\frac{1}{2}) + b'_+(\frac{1}{2}) b_+(\frac{1}{2})}{[b_+(\frac{1}{2}) + b_+(\frac{1}{2})]^2} = -2 \frac{-\frac{2}{(1/2)^3} - \frac{2}{(1/2)^3}}{2^2} = -16.
\end{aligned}$$

□

Definition 3.3.3. Define the function

$$b_1 : \mathbb{R} \to \mathbb{R}, \quad x \mapsto b_1(x) := \int_0^x b(t-1)dt.$$

Lemma 3.3.4. *(i)* $b_1(x) = x$ *for* $x \leq \frac{5}{4}$, *and*
(ii) $b_1(x) = \frac{3}{2}$ *for* $x \geq \frac{7}{4}$.

Proof. (i): If $x \geq \frac{7}{4}$, then

$$\begin{aligned}
\int_0^x b(t-1)dt &= \int_0^3 b(t-1)dt \\
&= \int_0^{\frac{3}{2}} b(t-1)dt + \int_{\frac{3}{2}}^3 b(t-1)dt \\
&= \int_3^{\frac{3}{2}} b(2-u)(-du) + \int_{\frac{3}{2}}^3 [1 - b(2-t)]dt \\
&= \frac{3}{2},
\end{aligned}$$

where we used the substitution $u = 3 - t$.
(ii):
If $x \leq \frac{5}{4}$, then

$$b_1(x) = \int_0^x b(t-1)dt = \int_0^x dt = x,$$

because $b(\xi) = 1$ for $\xi \leq \frac{1}{4}$. □

3.3.2 The Auxiliary Function C_N

In this subsection, we provide an auxiliary function C_N, in order to construct the function $\Gamma_N : \overline{\mathbb{D}} \to \mathbb{C}$ with the property $\Gamma_N|_{\partial\mathbb{D}} = \tilde{\gamma}$. During this subsection, let $\gamma : \mathbb{R} \to \mathbb{R}$ be a function which fulfills $\tilde{\gamma}(e^{it}) = e^{i\gamma(t)}$ for all $t \in \gamma$.

Definition 3.3.5 (Estimation constants)**.** We define $\eta_N : \mathbb{R}^2 \to \mathbb{C}$ by

$$\begin{aligned} \eta_N(x,y) &:= \frac{1}{y^2}[\frac{(iy)^2}{2}\gamma''(x) + \cdots + \frac{(iy)^N}{N!}\gamma^{(N)}(x)] \\ &= \alpha(x,y) + i\beta(x,y), \end{aligned}$$

such that we can define the estimation constants

$$\begin{aligned} s_0 &:= \mathsf{max}_{x\in\mathbb{R}}\,|\gamma'(x)|, \\ s_1 &:= \mathsf{max}_{x\in\mathbb{R}}\,|\gamma''(x)|, \\ s_2 &:= \mathsf{max}_{x\in\mathbb{R},y\in[-1,1]}\,|\eta_N(x,y)|, \\ s_3 &:= \mathsf{max}_{x\in\mathbb{R},y\in[-1,1]}\,|\partial_x\eta_N(x,y)|, \\ s_4 &:= \mathsf{max}_{x\in\mathbb{R},y\in[-1,1]}\,|\partial_y\eta_N(x,y)|, \end{aligned}$$

$$\begin{aligned} h_1 &:= \frac{1}{1+s_0+s_1+s_2+s_3+s_4}\cdot\frac{1}{100}\cdot\mathsf{min}_{x\in\mathbb{R}}|\gamma'(x)|, \\ h_2 &:= 100\cdot h_1\cdot\mathsf{max}_{x\in\mathbb{R}}(|\gamma'(x)|+1),\ \text{and} \\ h_3 &:= 100\cdot(1+h_2)\cdot\mathsf{max}_{x\in\mathbb{R}}(|\gamma(x)-x|+1). \end{aligned}$$

Definition 3.3.6 (Auxiliary function)**.** We define $C_N : \mathbb{R}^2 \to \mathbb{C}$ by

$$\begin{aligned} C_N(x,y) &:= x + (\gamma(x)-x)\cdot b(\frac{y}{h_3}) \\ &\quad +i[y + (\gamma'(x)-1)\cdot h_1\cdot b(\frac{y}{h_2})\cdot b_1(\frac{y}{h_1})] + y^2\cdot\eta_N(x,y)\cdot b(\frac{y}{h_1}), \end{aligned}$$

and $\tilde{C}_N : \mathbb{C} \to \mathbb{C}$ by $\tilde{C}_N(z) := \tilde{C}_N(\mathsf{Re}\,z, \mathsf{Im}\,z)$.

Lemma 3.3.7. *We have*

(i) $0 < h_1 < 1$*,*

(ii) $100\,h_1 \leq h_2$*,*

(iii) $100\,h_2 \leq h_3$*, and*

(iv) $\tilde{C}_N(z+2\pi) = \tilde{C}_N(z) + 2\pi$*.*

Remark: The following lemma points out that C_N deforms

$$\gamma(x) + (iy)\gamma'(x) + \frac{(iy)^2}{2}\gamma''(x) + \cdots + \frac{(iy)^N}{N!}\gamma^{(N)}(x)$$

via 3,5 and 9 into the identity map

$$x + iy$$

in a smooth manner.

Lemma 3.3.8. *The function*

$$C_N : \mathbb{R}^2 \to \mathbb{C}, \quad (x,y) \mapsto C_N(x,y)$$

on different parts of the domain is

1	$\gamma(x) + (iy)\gamma'(x) + y^2\eta_N(x,y)$	*for*	$y \leq \frac{1}{4}h_1,$
2	$\gamma(x) + (iy)\gamma'(x) + y^2\eta_N(x,y)b(\frac{y}{h_1})$	*for*	$\frac{1}{4}h_1 \leq y \leq \frac{3}{4}h_1,$
3	$\gamma(x) + (iy)\gamma'(x)$	*for*	$\frac{3}{4}h_1 \leq y \leq \frac{5}{4}h_1,$
4	$\gamma(x) + i[y + (\gamma'(x) - 1)h_1 \cdot b_1(\frac{y}{h_1})]$	*for*	$\frac{5}{4}h_1 \leq y \leq \frac{7}{4}h_1,$
5	$\gamma(x) + i[y + (\gamma'(x) - 1)\frac{3}{2}h_1]$	*for*	$\frac{7}{4}h_1 \leq y \leq \frac{1}{4}h_2,$
6	$\gamma(x) + i[y + (\gamma'(x) - 1)\frac{3}{2}h_1 \cdot b(\frac{y}{h_2})]$	*for*	$\frac{1}{4}h_2 \leq y \leq \frac{3}{4}h_2,$
7	$\gamma(x) + iy$	*for*	$\frac{3}{4}h_2 \leq y \leq \frac{1}{4}h_3,$
8	$x + (\gamma(x) - x)b(\frac{y}{h_3}) + iy$	*for*	$\frac{1}{4}h_3 \leq y \leq \frac{3}{4}h_3,$ *and*
9	$x + iy$	*for*	$\frac{3}{4}h_3 \leq y$.

Proof. Using Lemma 3.3.2(ii)+(iii) and Lemma 3.3.4 yields the result. $\square$

Lemma 3.3.9. *Let $x, y \in \mathbb{R}$ with $y \leq \frac{1}{4}h_1$, then*

(i) $C_N(x,y) = \gamma(x) + (iy)\gamma'(x) + \frac{(iy)^2}{2}\gamma''(x) + \cdots + \frac{(iy)^N}{N!}\gamma^{(N)}(x),$

(ii)

$$\begin{aligned} \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})C_N(x,y) &= \gamma'(x) + (iy)\gamma''(x) + \frac{(iy)^2}{2}\gamma'''(x) + \cdots \\ &+ \frac{(iy)^{(N-1)}}{(N-1)!}\gamma^{(N)}(x) + \frac{1}{2}\frac{(iy)^N}{N!}\gamma^{(N+1)}(x), \end{aligned}$$

(iii) $\frac{1}{2}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y})C_N(x,y)=\frac{1}{2}\frac{(iy)^N}{N!}\gamma^{(N+1)}(x).$

Lemma 3.3.10. *Let u be the real and v the imaginary part of C_N, i.e.,*

$$C_N(x,y)=u(x,y)+iv(x,y).$$

If $y>-h_1$, then
(i) $\partial_x u>0$,
(ii) $\partial_y v>0$,
(iii) $\partial_x u\geq 2|\partial_x v|$, *and*
(iv) $\partial_y v\geq 2|\partial_y u|$.

Proof. We distinguish 5 cases with respect to the variable y. For every case we will verify the conditions (i)-(iv) separately.
1. Case - $y\in[-h_1,h_1]$:
Let α be the real and β the imaginary part of η_N, i.e.,

$$\eta_N(x,y)=\alpha(x,y)+i\beta(x,y).$$

They fulfill $|\partial_x\alpha(x,y)|\leq s_3$ and $|\partial_x\beta(x,y)|\leq s_3$ since $|y|<1$. Moreover, we have

$$\begin{aligned}C_N(x,y) &:= \gamma(x)+iy\cdot\gamma'(x)+y^2\cdot\eta_N(x,y)\cdot b(\frac{y}{h_1})\\ &= u(x,y)+iv(x,y)\end{aligned}$$

with

$$\begin{aligned}u &= \gamma(x)+y^2\cdot\alpha(x,y)\cdot b(\frac{y}{h_1}), \qquad \text{and}\\ v &= y\cdot\gamma'(x)+y^2\cdot\beta(x,y)\cdot b(\frac{y}{h_1}).\end{aligned}$$

The four partial derivatives are

$$\begin{aligned}\partial_x u &= \gamma'(x)+y^2\cdot\partial_x\alpha(x,y)\cdot b(\frac{y}{h_1}),\\ \partial_x v &= y\cdot\gamma''(x)+y\cdot\partial_x\beta(x,y)\cdot b(\frac{y}{h_1}),\\ \partial_y u &= 2y\cdot\alpha(x,y)\cdot b(\frac{y}{h_1})+y^2\cdot\partial_y\alpha(x,y)\cdot b(\frac{y}{h_1})\\ &\quad +y^2\cdot\alpha(x,y)\cdot\frac{1}{h_1}\cdot b'(\frac{y}{h_1}),\\ \partial_y v &= \gamma'(x)+2y\cdot\beta(x,y)\cdot b(\frac{y}{h_1})+y^2\cdot\partial_y\beta(x,y)\cdot b(\frac{y}{h_1})\\ &\quad +y^2\cdot\beta(x,y)\cdot\frac{1}{h_1}\cdot b'(\frac{y}{h_1}).\end{aligned}$$

By the definition of h_1 in 3.3.5, we have

$$|y| \leq h_1 \quad < \quad \frac{\gamma'(x)}{s_3 + 2s_1 + 2s_3}$$

and therefore

$$\begin{aligned}
\gamma'(x) \quad &> \quad |y|(s_3 + 2s_1 + 2s_3) \\
&\geq \quad |y|(|\partial_x\alpha(x,y)| + 2|\gamma''(x)| + 2|\partial_x\beta(x,y)|) \\
&\geq \quad |y^2 \cdot \partial_x\alpha(x,y) \cdot b(\frac{y}{h_1})| + 2|y \cdot \gamma''(x) + y^2 \cdot \partial_x\beta(x,y) \cdot b(\frac{y}{h_1})| \\
&\geq \quad |y^2 \cdot \partial_x\alpha(x,y) \cdot b(\frac{y}{h_1})| + 2|\partial_x v|
\end{aligned}$$

using $|y| \leq h_1 < 1$ and $b(y/h_1) \leq 1$. Hence,

$$\partial_x u(x,y) = \gamma'(x) + y^2 \cdot \partial_x\alpha(x,y) \cdot b(\frac{y}{h_1}) > 2|\partial_x v|,$$

and we have shown (iii). If we take into account that the inequality above is a proper one, we see that (i) holds. The inequality

$$|y| \leq h_1 < \frac{\gamma'(x)}{2(2s_2 + s_4 + 16s_2) + 2(2s_2 + s_4 + 16s_2)}$$

implies

$$\begin{aligned}
&\gamma'(x) \\
> \quad & |y|(2(2s_2 + s_4 + 16s_2) + 2(2s_2 + s_4 + 16s_2)) \\
\geq \quad & 2|y|(|2\beta(x,y)| + |\partial_y\beta(x,y)| + 16|\beta(x,y)|) \\
& +2|y|(|2\alpha(x,y)| + |\partial_y\alpha(x,y)| + 16|\alpha(x,y)|) \\
\geq \quad & 2\,|2y \cdot \beta(x,y) \cdot b(\frac{y}{h_1}) + y^2 \cdot \partial_y\beta(x,y) \cdot b(\frac{y}{h_1}) + y^2 \cdot \beta(x,y) \cdot \frac{1}{h_1} \cdot b'(\frac{y}{h_1})| \\
& +2\,|2y \cdot \alpha(x,y) \cdot b(\frac{y}{h_1}) + y^2 \cdot \partial_y\alpha(x,y) \cdot b(\frac{y}{h_1}) + y^2 \cdot \alpha(x,y) \cdot \frac{1}{h_1} \cdot b'(\frac{y}{h_1})| \\
\geq \quad & 2 \cdot |\partial_y v - \gamma'(x)| + 2|\partial_y u| \\
\geq \quad & |\partial_y v - \gamma'(x)| + 2|\partial_y u| \\
\geq \quad & -\partial_y v + \gamma'(x) + 2|\partial_y u|,
\end{aligned}$$

and therefore $\partial_y v > 2|\partial_y u|$ and (iv) is proved. Moreover, this yields $\partial_y v > 0$, and hence (ii) is proved.

2. Case - $y \in [h_1, 2h_1]$:
Here we have to consider

$$C_N(x,y) = \underbrace{\gamma(x)}_{u(x,y)} + i \underbrace{[y + (\gamma'(x) - 1)\, h_1\, b_1(\frac{y}{h_1})]}_{v(x,y)}.$$

The partial derivatives are

$$\begin{aligned} \partial_x u &= \gamma'(x), \\ \partial_x v &= \gamma''(x) h_1 \cdot b_1(\frac{y}{h_1}), \\ \partial_y u &= 0, \qquad \text{and} \\ \partial_y v &= 1 + \underbrace{(\gamma'(x) - 1)}_{>-1} \cdot \underbrace{b_1'(\frac{y}{h_1})}_{\leq 1}. \end{aligned}$$

The conditions (i) and (ii) are satisfied due to $\gamma'(x) > 0$, and condition (iv) is obvious. It remains to check condition (iii) for this case. By the definition of h_1 and s_1, we have

$$h_1 \leq \frac{1}{3}\frac{|\gamma'(x)|}{s_1} \leq \frac{1}{3}\frac{|\gamma'(x)|}{|\gamma''(x)|}$$

for all $x \in \mathbb{R}$. Since $b_1(y/h_1) \leq 3/2$, we get

$$2|\gamma''(x)| \cdot h_1 \cdot b_1(\frac{y}{h_1}) \leq |\gamma'(x)|$$

by Definition 3.3.3, and conclude

$$2|\partial_x v| = 2|\gamma''(x)| \cdot h_1 \cdot b_1(\frac{y}{h_1}) \leq |\gamma'(x)| = |\partial_x u|.$$

3. Case - $y \in [2h_1, h_2]$:
For

$$C_N(x,y) = \underbrace{\gamma(x)}_{u(x,y)} + i \underbrace{[y + (\gamma'(x) - 1)\frac{3}{2} h_1 b(\frac{y}{h_2})]}_{v(x,y)},$$

the partial derivatives are

$$\begin{aligned} \partial_x u &= \gamma'(x), \\ \partial_x v &= \gamma''(x) \frac{3}{2} h_1 \cdot \underbrace{b(\frac{y}{h_2})}_{\leq 1}, \\ \partial_y u &= 0, \ \text{and} \\ \partial_y v &= 1 + (\gamma'(x) - 1)\frac{3}{2} \cdot h_1 \cdot \frac{1}{h_2} \cdot b'(\frac{y}{h_2})]. \end{aligned}$$

We have $\gamma'(x) > 0$, because γ is a diffeomorphism. This implies $\partial_x u > 0$ and the condition (i) is verified. Now we will prove (ii). Since

$$h_2 = 100 h_1 \mathsf{max}_{t\in\mathbb{R}}(|\gamma'(t)|+1)$$

and $|b'(y/h_2)| \leq 16$, we have

$$(|\gamma'(x)|+1)\frac{3}{2}\frac{h_1}{h_2}|b'(\frac{y}{h_2})| < \frac{3}{2}\,\frac{1}{100}\,16 < \frac{1}{2}.$$

This implies

$$\begin{aligned}
\partial_y v &= 1 + (\gamma'(x)-1)\frac{3}{2}h_1\frac{1}{h_2}b'(\frac{y}{h_2}) \\
&\geq 1 - (|\gamma'(x)|+1)\frac{3}{2}\frac{h_1}{h_2}|b'(\frac{y}{h_2})| \\
&\geq 1 - \frac{1}{2} > 0,
\end{aligned}$$

taking $b'(y/h_2) < 0$ into account. Hence, condition (ii) is fulfilled. Condition (iv) follows from $\partial_y u = 0$. Next, we will prove condition (iii). By the definition of h_1 and s_1, we have

$$|\frac{1}{3}\frac{\gamma'(x)}{\gamma''(x)}| \geq |\frac{1}{3}\frac{\gamma'(x)}{s_1}| \geq h_1$$

and compute

$$\begin{aligned}
\partial_x u &= \gamma'(x) \geq |\gamma''(x)|\cdot 3|\frac{1}{3}\frac{\gamma'(x)}{\gamma''(x)}|\cdot b(\frac{y}{h_2}) \geq 2|\gamma''(x)\cdot\frac{3}{2}h_1\cdot b(\frac{y}{h_2})| \\
&= 2|\partial_x v|,
\end{aligned}$$

which is equivalent to condition (iii).
4. Case - $y \in [h_2, h_3]$:
For

$$C_N(x,y) = \underbrace{x + (\gamma(x)-x)\cdot b(\frac{y}{h_3})}_{=u} + i\underbrace{y}_{=v}$$

the partial derivatives are

$$\begin{aligned}
\partial_x u &= 1 + (\gamma(x)'-1)\cdot b(\frac{y}{h_3}), \\
\partial_x v &= 0, \\
\partial_y u &= (\gamma(x)-x)\cdot\frac{1}{h_3}\cdot b'(\frac{y}{h_3}), \text{ and} \\
\partial_y v &= 1.
\end{aligned}$$

The equation

$$\partial_x u = 1 + \underbrace{(\gamma'(x) - 1)}_{\in(-1,\infty)} \cdot \underbrace{b(\frac{y}{h_3})}_{\in[0,1]}$$

proves condition (i). By the partial derivative $\partial_y v = 1$, condition (ii) is obvious and by $\partial_x v = 0$ condition (iii) is obvious. It remains to show condition (iv). By the definition of h_3, we have

$$h_3 \quad := \quad 100 \cdot (1 + h_2) \cdot \mathsf{max}_{x \in \mathbb{R}}(|\gamma(x) - x| + 1) > 100|\gamma(x) - x|,$$

which implies

$$\begin{aligned} \partial_y v = 1 \quad &\geq \quad 100\frac{|\gamma(x) - x|}{h_3} \\ &\geq \quad 2|\gamma(x) - x|\frac{1}{h_3}\underbrace{|b'(\frac{y}{h_3})|}_{\leq 16} \\ &\geq \quad 2|(\gamma(x) - x)\frac{1}{h_3}b'(\frac{y}{h_3})| = 2|\partial_y u|. \end{aligned}$$

5. Case - $y \geq h_3$:
Finally

$$C_N(x, y) := x + iy = u + iv,$$

yields $\partial_x u = 1$, and $\partial_y v = 1$, which implies condition (i) and (ii). On the other hand $\partial_x v = 0$ and $\partial_y u = 0$ holds, and we obtain condition (iii) and (iv). □

Lemma 3.3.11. *Consider the upper half plane*

$$H = \{\, (x, y) \in \mathbb{R} \,:\, y \geq 0 \,\},$$

and let $f : H \to \mathbb{R}^2, \quad (x, y) \mapsto (u(x, y), v(x, y))$ *be a smooth function. If*

(i) $\partial_x u > 0$,
(ii) $\partial_y v > 0$,
(iii) $\partial_x u \geq 2|\partial_x v|$, *and*
(iv) $\partial_y v \geq 2|\partial_y u|$,
then f *is injective.*

Proof. Let $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ be two points in H such that

$$f(x_1, y_1) = f(x_2, y_2).$$

Assume $x_2 > x_1$. The condition $\partial_x u > 0$ implies

$$|u(x_2, y_1) - u(x_1, y_1)| > 0. \qquad (\dagger)$$

Moreover, $\partial_x u > 2|\partial_x v|$ has the consequence

$$|u(x_2, y_1) - u(x_1, y_1)| = |\int_{x_1}^{x_2} \partial_x u(x, y_1) dx| = \int_{x_1}^{x_2} |\partial_x u(x, y_1)| dx$$
$$\geq 2 \int_{x_1}^{x_2} |\partial_x v(x, y_1)| dx \geq 2| \int_{x_1}^{x_2} \partial_x v(x, y_1) dx| = 2|v(x_2, y_1) - v(x_1, y_1)|$$

and similarly $\partial_y v > 2|\partial_y u|$ is the reason for

$$|v(x_2, y_1) - v(x_2, y_2)| \geq 2|u(x_2, y_1) - u(x_2, y_2)|.$$

We obtain

$$|u(x_2, y_1) - u(x_1, y_1)| \geq 4|u(x_2, y_1) - u(x_1, y_1)|$$

using $u(x_2, y_2) = u(x_1, y_1)$ and $v(x_2, y_2) = v(x_1, y_1)$. This shows

$$|u(x_2, y_1) - u(x_1, y_1)| = 0,$$

which is a contradiction to $(\dagger)$. This contradiction yields $x_1 = x_2$. The argumentation for $y_1 = y_2$ can be performed in the same way by interchanging u with v. We conclude that the two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ are the same, and hence that f is injective on H. $\square$

Lemma 3.3.12. *Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be a continuously differentiable function and*

$$f(x + iy) = u(x, y) + iv(x, y)$$

the decomposition in its real and imaginary part. If
(i) $\partial_x u > 0$,
(ii) $\partial_y v > 0$,
(iii) $\partial_x u \geq 2|\partial_x v|$, *and*
(iv) $\partial_y v \geq 2|\partial_y u|$,
then $\partial_z f \neq 0$, *and* $|\partial_{\bar{z}} f| < |\partial_z f|$.

Proof. The partial derivatives with respect to the complex variables

$$z = x + iy \qquad \text{and} \qquad \bar{z} = x - iy$$

are

$$\begin{aligned}\partial_z f &= \frac{1}{2}(\partial_x - i\partial_y)(u+iv) = \frac{1}{2}(u_x + v_y) + \frac{i}{2}(v_x - u_y), \text{ and}\\ \partial_{\bar{z}} f &= \frac{1}{2}(\partial_x + i\partial_y)(u+iv) = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y).\end{aligned}$$

The first two assumptions $u_x > 0$ and $v_y > 0$ yield $\partial_z f \neq 0$. Moreover, we have $u_x \cdot v_y > 0$ due to (i) and (ii), and

$$u_x \cdot v_y \geq 4|v_x| \cdot |u_y| \geq 4 \cdot v_x \cdot u_y$$

by the assumptions (iii) and (iv). Adding these two inequalities results in

$$4u_x \cdot v_y > 4v_x \cdot u_y.$$

This is equivalent to

$$(u_x + v_y)^2 + (v_x - u_y)^2 > (u_x - v_y)^2 + (v_x + u_y)^2,$$

which implies

$$|\partial_{\bar{z}} f| = |(u_x - v_y) + i(v_x + u_y)| < |(u_x + v_y) + i(v_x - u_y)| = |\partial_z f|.$$

□

Corollary 3.3.13. *Let $z \in \mathbb{C}$ with $\mathsf{Im}\, z > -h_1$, then*
(i) $(\partial_z \tilde{C})(z) \neq 0$, *and*
(ii) $(\partial_{\bar{z}} \tilde{C})(z) < (\partial_z \tilde{C})(z)$.

Proof. This is the combination of Lemma 3.3.10 and Lemma 3.3.12. □

3.3.3 The Auxiliary Function Γ_N

Definition 3.3.14. Define the function

$$\Gamma_N : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \Gamma_N(z) := \begin{cases} e^{i\tilde{C}_N(\frac{1}{i}\mathsf{log}\, z)} & | \quad z \neq 0 \\ 0 & | \quad z = 0. \end{cases}$$

Here one use any definition of the logarithm as far as $\mathsf{exp}(\mathsf{log}(z)) = z$ is fulfilled, because the ambiguity of the logarithm is absorbed by the exponential map due to $\tilde{C}_N(z + 2\pi) = 2\pi + \tilde{C}_N(z)$.

Lemma 3.3.15. *There exists a* 0*-neighborhood* $U \subseteq \mathbb{C}$ *such that the restriction of* Γ_N *to* U *is the identity map.*

Proof. Let $U := \{ z \in \mathbb{C} : |z| < \mathsf{exp}(-h_3) \}$ with h_3 of Definition 3.3.5. We show that $\Gamma|_U$ is the identity map. If $|z| < \mathsf{exp}(-h_3)$, then $-\mathsf{log}|z| > h_3$ and therefore

$$\begin{aligned}
\Gamma_N(z) &= \mathsf{exp}\, i\, \tilde{C}_N(\frac{1}{i}\mathsf{log}\, z) \\
&= \mathsf{exp}\, i\, C_N(\mathsf{Re}(\frac{1}{i}\mathsf{log}\, z), \underbrace{\mathsf{Im}(\frac{1}{i}\mathsf{log}\, z)}_{-\mathsf{log}|z|}) \\
&= \mathsf{exp}\, i\, (\mathsf{Re}(\frac{1}{i}\mathsf{log}\, z) + i\mathsf{Im}(\frac{1}{i}\mathsf{log}\, z)) \\
&= z.
\end{aligned}$$

In this computation we have used Lemma 3.3.8, which states that

$$C_N(x, y) = x + iy$$

for $y > h_3$. $\square$

Lemma 3.3.16. $\Gamma_N : \mathbb{C} \to \mathbb{C}$ *is smooth.*

Proof. We have only to consider the point $z = 0$. On the rest of the complex plane the function Γ_N is smooth by construction. But in a neighborhood U of $z = 0$ the function Γ_N is the identity map, due to Lemma 3.3.15. $\square$

Lemma 3.3.17. *The derivatives of* $\Gamma_N : \mathbb{C} \to \mathbb{C}$ *are*

(i) $\partial_{\bar{z}}\Gamma_N(z) = -\frac{1}{\bar{z}} \cdot \Gamma_N(z)(\partial_{\bar{z}}\tilde{C}_N)(\frac{1}{i}\mathsf{log}\, z)$, *and*

(ii) $\partial_z\Gamma_N(z) = \frac{1}{z} \cdot \Gamma_N(z)(\partial_z\tilde{C}_N)(\frac{1}{i}\mathsf{log}\, z)$.

Proof. Abbreviate $\xi(z) := -i\,\mathsf{log}\, z$, then we have $\overline{\xi(z)} = i\,\mathsf{log}\,\bar{z}$ and $\Gamma_N(z) = e^{i\tilde{C}_N(\xi(z))}$. With the complex chain rules

$$\begin{aligned}
\partial_z(f \circ g)(z) &= (\partial_z f)(g(z)) \cdot \partial_z g(z) + (\partial_{\bar{z}} f)(g(z)) \cdot \partial_z \overline{g(z)}, \text{ and} \\
\partial_{\bar{z}}(f \circ g)(z) &= (\partial_z f)(g(z)) \cdot \partial_{\bar{z}} g(z) + (\partial_{\bar{z}} f)(g(z)) \cdot \partial_{\bar{z}} \overline{g(z)}
\end{aligned}$$

we perform the following computations.

(i):

$$\begin{aligned}
\partial_{\bar z}\Gamma_N(z) &= i\,\underbrace{e^{i\tilde C_N(\xi(z))}}_{\Gamma_N(z)}\cdot\partial_{\bar z}(\tilde C_N\circ\xi)(z)\\
&= i\Gamma_N(z)[(\partial_z\tilde C_N)(\xi(z))\cdot\underbrace{\partial_{\bar z}\xi(z)}_{=0}+(\partial_{\bar z}\tilde C_N)(\xi(z))\cdot\underbrace{\partial_{\bar z}\overline{\xi(z)}}_{=\frac{i}{\bar z}}]\\
&= -\frac{1}{\bar z}\cdot\Gamma_N(z)(\partial_{\bar z}\tilde C_N)(\xi(z))\\
&= -\frac{1}{\bar z}\cdot\Gamma_N(z)(\partial_{\bar z}\tilde C_N)(\frac{1}{i}\mathsf{log}\, z).
\end{aligned}$$

(ii):

$$\begin{aligned}
\partial_{z}\Gamma_N(z) &= ie^{i\tilde C_N(\xi(z))}\cdot\partial_{z}(\tilde C_N\circ\xi)(z)\\
&= i\Gamma_N(z)[(\partial_z\tilde C_N)(\xi(z))\cdot\underbrace{\partial_{z}\xi(z)}_{=-\frac{i}{z}}+(\partial_{\bar z}\tilde C_N)(\xi(z))\cdot\underbrace{\partial_{z}(\overline{\xi(z)})}_{=0}]\\
&= \frac{1}{z}\cdot\Gamma_N(z)\cdot(\partial_{z}\tilde C_N)(\xi(z))\\
&= \frac{1}{z}\cdot\Gamma_N(z)(\partial_{z}\tilde C_N)(\frac{1}{i}\mathsf{log}\, z).
\end{aligned}$$

□

Lemma 3.3.18. *We have*

$$\left|\frac{\partial_{\bar z}\Gamma_N(z)}{\partial_z\Gamma_N(z)}\right|<1$$

for $z\in\overline{\mathbb{D}}$.

Proof. It follows from Corollary 3.3.13(ii) that

$$\left|\frac{\partial_{\bar\xi}\tilde C_N(\xi)}{\partial_{\xi}\tilde C_N(\xi)}\right|<1,$$

for $\xi\in\mathbb{C}$ with $\mathsf{Im}\,\xi\geq 0$. Lemma 3.3.17 yields

$$-\frac{\partial_{\bar z}\Gamma_N(z)}{\partial_z\Gamma_N(z)}=\frac{z}{\bar z}\cdot\frac{(\partial_{\bar z}\tilde C_N)(\frac{1}{i}\mathsf{log}\, z)}{(\partial_z C_N)(\frac{1}{i}\mathsf{log}\, z)}.$$

Since $\mathsf{Im}(-i\cdot\mathsf{log}\, z)=-\mathsf{Re}(\mathsf{log}\, z)\geq 0$ for $z\in\overline{\mathbb{D}}$, we conclude

$$|\frac{\partial_{\bar z}\Gamma_N(z)}{\partial_z\Gamma_N(z)}|=|\frac{z}{\bar z}|\cdot|\frac{(\partial_{\bar z}\tilde C_N)(\frac{1}{i}\mathsf{log}\, z)}{(\partial_z C_N)(\frac{1}{i}\mathsf{log}\, z)}|<1$$

for all $z\in\overline{\mathbb{D}}$. □

Lemma 3.3.19. *If $z = e^{it}$ with $t \in \mathbb{R}$, then*

(i) $\Gamma_N(z) = \tilde{\gamma}(z)$,

(ii) $(\partial_{\bar{z}}\Gamma_N)(z) = 0$, *and*

(iii) $(\partial_z\Gamma_N)(z) = e^{i(\gamma(t)-t)} \cdot \gamma'(t)$.

Proof. (i):

$$\Gamma_N(e^{it}) \overset{3.3.14}{=} e^{i\tilde{C}_N(\frac{1}{i}\log e^{it})} = e^{i\tilde{C}_N(t+i\cdot 0)} \overset{3.3.8}{=} e^{i\gamma(t)} = \tilde{\gamma}(e^{it}).$$

(ii): By Lemma 3.3.17(i), we have

$$\partial_{\bar{z}}\Gamma_N(z) = -\frac{1}{\bar{z}} \cdot \Gamma_N(z)(\partial_{\bar{z}}\tilde{C}_N)(\frac{1}{i}\log z)$$

and by Lemma 3.3.9(iii)

$$\partial_{\bar{z}}\tilde{C}_N(z) = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})C_N(x,y) = \frac{1}{2}\frac{(iy)^N}{N!}\gamma^{(N+1)}(x)$$

with $z = x + iy$. The substitution $z = e^{it}$ yields

$$(\partial_{\bar{z}}\Gamma_N)(e^{it}) = -e^{it}\Gamma_N(e^{it}) \cdot (\partial_{\bar{z}}\tilde{C}_N)(t + i0) = 0.$$

(iii): Lemma 3.3.17(ii) states

$$\partial_z\Gamma_N(z) = \frac{1}{z} \cdot \Gamma_N(z) \cdot (\partial_z\tilde{C}_N)(\frac{1}{i}\log z)$$

and with Lemma 3.3.9(ii)

$$\begin{aligned}
& \partial_z\tilde{C}_N \\
= \; & \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})C_N(x,y) \\
= \; & \gamma'(x) + (iy)\gamma''(x) + \cdots + \frac{(iy)^{(N-1)}}{(N-1)!}\gamma^{(N)}(x) + \frac{1}{2}\frac{(iy)^N}{N!}\gamma^{(N+1)}(x)
\end{aligned}$$

we get

$$\begin{aligned}
(\partial_z\Gamma_N)(e^{it}) &= \frac{1}{z}\Gamma_N(z) \cdot (\partial_z\tilde{C}_N)(t + i0) \\
&= e^{-it}\Gamma_N(e^{it}) \cdot [\gamma'(t) + 0 + \cdots + 0] \\
&= e^{-it}\tilde{\gamma}(e^{it})\gamma'(t) \\
&= e^{i(\gamma(t)-t)}\gamma'(t).
\end{aligned}$$

$\square$

Lemma 3.3.20. *All partial derivative of the complex dilatation*

$$\mu_N(z) := \frac{\partial_{\bar z}\Gamma_N(z)}{\partial_z\Gamma_N(z)}$$

up to order $N-1$, vanish on the boundary $\partial\mathbb{D}$ of $\mathbb{D}$.

Proof. By Lemma 3.3.8, we have

$$C_N(x,y) = \gamma(x) + (iy)\gamma'(x) + \underbrace{\frac{(iy)^2}{2}\gamma''(x) + \cdots + \frac{(iy)^N}{N!}\gamma^{(N)}(x)}_{=y^2\cdot\eta(x,y)}$$

for $|y| \le \frac{1}{4}h_1$. Moreover, by Lemma 3.3.9(iii), the derivative with respect to $\bar z$ is

$$\begin{aligned}(\partial_{\bar z}\tilde C_N)(z) &= \frac{1}{2}(\partial_x + i\partial_y)C_N(x,y) \qquad [z = x+iy] \\ &= \frac{1}{2}\frac{(iy)^N}{N!}\tilde\gamma^{(N+1)}(x)\end{aligned}$$

for $z \in \mathbb{C}$ with $|\mathsf{Im}\, z| \le \frac{1}{4}h_1$. Hence, we see that for $\mathsf{Im}\, z = 0$ all partial derivatives of $(\partial_{\bar z}\tilde C_N)(z)$ up to order $(N-1)$ are equal to zero. Corollary 3.3.13(i) asserts that $(\partial_z\tilde C_N)(z) \neq 0$ for $\mathsf{Im}\, z > -h_1$. Taking into account that $(\partial_{\bar z}\tilde C_N)(z)$ is smooth, we obtain that for $\mathsf{Im}\, z = 0$ all partial derivatives of

$$\frac{(\partial_{\bar z}\tilde C_N)(z)}{(\partial_z\tilde C_N)(z)} \qquad \text{[Leibniz rule]}$$

up to order $(N-1)$ are equal to zero. Finally, we conclude that for $\xi \in \mathbb{C}$ with $|\xi| = 1$ all partial derivatives of

$$\mu_N(\xi) = \frac{\partial_{\bar\xi}\Gamma_N(\xi)}{\partial_\xi\Gamma_N(\xi)} = -\frac{\xi}{\bar\xi}\cdot\frac{(\partial_{\bar z}\tilde C_N)(\frac{1}{i}\log\xi)}{(\partial_z\tilde C_N)(\frac{1}{i}\log\xi)}$$

up to order $(N-1)$ vanish are equal to zero. ☐

Lemma 3.3.21. *Let $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be a continuous function which is injective, and the restriction to the boundary $f|_{\partial\mathbb{D}} : \partial\mathbb{D} \to \partial\mathbb{D}$ is a homeomorphism. Then $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ itself is a homeomorphism.*

Proof. On a compact Hausdorff space there exists no coarser Hausdorff topology. This shows that $f : \overline{\mathbb{D}} \to f(\overline{\mathbb{D}})$ is a homeomorphism. It remains to show that $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is surjective. Assume this is not the case, then there exists an element $x_0 \in \overline{\mathbb{D}}$ with

$$x_0 \notin f(\mathbb{D}).$$

Consider the function

$$\mathbb{S}^1 \to \partial\mathbb{D}, \quad t \to e^{it}.$$

Its winding number with respect to x_0 is non-zero, because $|x_0| < 1$. This implies that $f(\mathbb{D})$ is not contractible. Since $f : \overline{\mathbb{D}} \to f(\overline{\mathbb{D}})$ is a homeomorphism, $f(\overline{\mathbb{D}})$ must be a contractible, which is a contradiction. We conclude that $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is surjective and hence a homeomorphism. $\square$

Lemma 3.3.22. $\Gamma_N : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ *is a homeomorphism.*

Proof. First, we will show that Γ_N is injective on $\overline{\mathbb{D}}$. By Lemma 3.3.10 and 3.3.11, the function

$$\tilde{C}_N : H \to H$$

is injective, where $H \subseteq \mathbb{C}$ is the upper half plane. Since

$$\tilde{C}_N(z + 2\pi) = 2\pi + \tilde{C}_N(z)$$

we see that

$$\Gamma_N : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \Gamma_N(z) := \begin{cases} e^{i\tilde{C}_N(\frac{1}{i}\log z)} & | \quad z \neq 0 \\ 0 & | \quad z = 0. \end{cases}$$

is injective. The function Γ_N is continuous by Lemma 3.3.16. Furthermore, $\Gamma_N|_{\partial\mathbb{D}} = \tilde{\gamma}$, which means that the restriction of Γ_N to the boundary of $\partial\mathbb{D}$ is a homeomorphism. Applying Lemma 3.3.21 yields that Γ_N is a homeomorphism on $\overline{\mathbb{D}}$. $\square$

Lemma 3.3.23. $\Gamma_N : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ *is a diffeomorphism.*

Proof. Lemma 3.3.18 asserts $|\partial_{\bar{z}}\Gamma_N(z)/\partial_z\Gamma_N(z)| < 1$ for $z \in \overline{\mathbb{D}}$. Due to Lemma 3.2.9, this is equivalent to the statement that the Jacobian determinant $\det d\Gamma_N$ is positive. Since Γ_N is smooth by Lemma 3.3.16, and a homeomorphism by Lemma 3.3.22, we conclude that Γ_N is a diffeomorphism. $\square$

Proposition 3.3.24. *Let $\tilde{\gamma} : \partial\mathbb{D} \to \partial\mathbb{D}$ be a diffeomorphism with winding number 1. Then there exists for every natural number $N \geq 2$ a diffeomorphism*

$$\Gamma_N : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$$

such that

(i) $\Gamma_N(0) = 0$, $(\partial_z \Gamma_N)(0) = 1$, $\Gamma_N|_{\partial\mathbb{D}} = \tilde{\gamma}$,

(ii) $(\partial_{\bar{z}} \Gamma_N)(e^{it}) = 0$, $(\partial_z \Gamma_N)(e^{it}) = e^{i(\gamma(t)-t)}\gamma'(t)$ *for* $t \in \mathbb{R}$, *with the diffeomorphism* $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$ *which fulfills* $e^{i\gamma(t)} = \tilde{\gamma}(e^{it})$,

(iii) *the function*

$$\mu_N : \overline{\mathbb{C}} \to \mathbb{C}, \quad z \to \left\{ \begin{array}{r|l} \frac{\partial_{\bar{z}}\Gamma_N(z)}{\partial_z \Gamma_N(z)} & z \in \mathbb{D} \\ 0 & z \in \overline{\mathbb{C}} \setminus \mathbb{D} \end{array} \right.$$

is $(N-1)$-times continuously differentiable, and

(iv) $\|\mu_N\|_\infty < 1$.

Proof. We will show that the function Γ_N defined in 3.3.14 satisfies the requirements. By Lemma 3.3.23, the function Γ_N is a diffeomorphism.
(i): By Lemma 3.3.15, there exists a zero-neighborhood $U \subseteq \mathbb{C}$, such that the restriction of Γ_N to U is the identity map. This yields $\Gamma_N(0) = 0$ and $(\partial_z \Gamma_N)(0) = 1$. Moreover, Lemma 3.3.19(i) asserts $\Gamma_N|_{\partial\mathbb{D}} \equiv \tilde{\gamma}$.
(ii): This is the statement of Lemma 3.3.19(ii)+(iii).
(iii): By Lemma 3.3.20, all partial derivative of the complex dilatation

$$\mu_N(z) := \frac{\partial_{\bar{z}}\Gamma_N(z)}{\partial_z \Gamma_N(z)}$$

up to order $(N-1)$, vanish on the boundary $\partial\mathbb{D}$.
(iv): This is a consequence of Lemma 3.3.18. □

3.4 Proof of the Theorem

In this section, we provide the necessary definitions to formulate the Birkhoff decomposition as a theorem, and give a proof. This proof is based on the results of the previous two sections.

Theorem 3.4.1 (Birkhoff Decomposition)**.** *For each $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$ there exist two functions $f \in \mathsf{V}^+$, $g \in \mathsf{V}^-$, and a complex number $c \in \mathbb{C}^\times$ such that*

$$f(\gamma(t)) = c \cdot g(t) \qquad (i)$$

for all $t \in \mathbb{S}^1$. Furthermore, g, f and c are unique.

Proof. Let $\tilde{\gamma} : \partial\mathbb{D} \to \partial\mathbb{D}$ be the diffeomorphism which fulfills

$$\tilde{\gamma}(e^{it}) = e^{\gamma(t)}.$$

By Proposition 3.3.24, there exists a sequence $(\Gamma_N)_{N\in\mathbb{N}}$ of diffeomorphisms $\Gamma_N : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that

(i) $\Gamma_N(0) = 0$, $\Gamma'_N(0) = 1$, $\Gamma_N|_{\partial\mathbb{D}} = \tilde{\gamma}$,

(ii) $(\partial_{\bar{z}}\Gamma_N)(e^{it}) = 0$, $(\partial_z\Gamma_N)(e^{it}) = e^{i(\gamma(t)-t)}\,\gamma'(t) \neq 0$ for $t \in \mathbb{R}$,

(iii) the function

$$\mu_N : \overline{\mathbb{C}} \to \mathbb{C}, \quad z \to \left\{ \begin{array}{r|l} \frac{\partial_{\bar{z}}\Gamma_N(z)}{\partial_z\Gamma_N(z)} & z \in \mathbb{D} \\ 0 & z \in \overline{\mathbb{C}}\backslash\mathbb{D} \end{array} \right.$$

is $(N+5)$-times continuously differentiable, and

(iv) $\|\mu_N\|_\infty < 1$ holds.

For every $N \in \mathbb{N}$ there exists by Proposition 3.2.18 a $\mathcal{C}^N$-isomorphism

$$\alpha_N : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$$

such that $\partial_{\bar{z}}\alpha_N(z) = \mu_N(z) \cdot \partial_z\alpha_N(z)$, $\alpha_N(0) = 0$, $\alpha'_N(0) = 1$, and that $\alpha_N(\infty) = \infty$ holds.

1. Step:

Since $\mu_N(z) = 0$ for $z \in \overline{\mathbb{C}}\backslash\mathbb{D}$ we have $\partial_{\bar{z}}\alpha_N \equiv 0$ on $\overline{\mathbb{C}}\backslash\overline{\mathbb{D}}$, and therefore the function

$$\alpha_N : \overline{\mathbb{C}}\backslash\overline{\mathbb{D}} \to \overline{\mathbb{C}}$$

is univalent. By Lemma 3.2.10,

$$\alpha_N \circ \Gamma_N^{-1} : \mathbb{D} \to \overline{\mathbb{C}}$$

is holomorphic, because $\partial_{\bar{z}}\alpha_N = \mu_N\,\partial_z\alpha_N$ and $\partial_{\bar{z}}\Gamma_N = \mu_N\,\partial_z\Gamma_N$. Moreover, since Γ_N and α_N are $\mathcal{C}^1$-isomorphisms, the map $\alpha_N \circ \Gamma_N^{-1} : \mathbb{D} \to \mathbb{C}$ is univalent.

2. Step:

For every $N, M \in \mathbb{N}$ define

$$\eta_{NM} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}, \quad z \to \left\{ \begin{array}{r|l} (\Gamma_N^{-1} \circ \Gamma_M)(z) & z \in \mathbb{D} \\ z & z \in \overline{\mathbb{C}}\backslash\mathbb{D} \end{array} \right. ,$$

which fulfills $\eta_{NM}(0) = 0$, $\eta'_{NM}(0) = 1$, and $\eta_{NM}(\infty) = \infty$.

Since $\Gamma_N|_{\partial\mathbb{D}} \equiv \Gamma_M|_{\partial\mathbb{D}}$, we obtain that η_{NM} is continuous. Due to

$$\partial_{\bar{z}}\Gamma_N|_{\partial\mathbb{D}} \equiv \partial_{\bar{z}}\Gamma_M|_{\partial\mathbb{D}} \equiv 0 \qquad \text{and} \qquad \partial_z\Gamma_N|_{\partial\mathbb{D}} \equiv \partial_z\Gamma_M|_{\partial\mathbb{D}}$$

we conclude that the derivatives of η_{NM} are continuous, and therefore η_{NM} itself is $\mathcal{C}^1$. Because the inverse of η_{NM} is η_{MN}, the inverse of η_{NM} is also $\mathcal{C}^1$, and we conclude that $\eta_{NM} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a $\mathcal{C}^1$-isomorphism.

3. Step: $\varphi := \alpha_N \circ \eta_{NM} \circ \alpha_M^{-1} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is the identity map:
Let $K_M := (\alpha_M \circ \Gamma_M^{-1})(\mathbb{D}) = \alpha_M(\mathbb{D})$ and $\partial K_M = (\alpha_M \circ \Gamma_M^{-1})(\partial\mathbb{D}) = \alpha_M(\partial\mathbb{D})$ be its boundary. Since $\alpha_N \circ \Gamma_N^{-1}$ and $\alpha_M \circ \Gamma_M^{-1}$ are univalent on $\mathbb{D}$, the restriction

$$\varphi|_{K_M} = \alpha_N \circ \Gamma_N^{-1} \circ (\alpha_M \circ \Gamma_M^{-1})^{-1}$$

is univalent too. On the other hand, α_N and α_M are univalent on $\overline{\mathbb{C}}\backslash\overline{\mathbb{D}}$ and therefore the restriction

$$\varphi|_{\overline{\mathbb{C}}\backslash\overline{K}_M} = \alpha_N \circ \alpha_M^{-1}$$

is also univalent, due to $\alpha_M^{-1}(\overline{\mathbb{C}}\backslash\overline{K}_M) = \overline{\mathbb{C}}\backslash\mathbb{D}$. We conclude that φ is univalent on $\overline{\mathbb{C}}\backslash\overline{K}_M$, and in particular holomorphic. This yields

$$\partial_{\bar{z}}\varphi|_{\overline{\mathbb{C}}\backslash\partial K_M} \equiv 0.$$

Since $\alpha_N, \alpha_M, \eta_{NM} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ are $\mathcal{C}^1$-isomorphisms, the composition φ is also a $\mathcal{C}^1$-isomorphism. Hence, $\partial_{\bar{z}}\varphi$ is continuous on $\overline{\mathbb{C}}$, which implies $\partial_{\bar{z}}\varphi \equiv 0$ on the whole Riemann sphere $\overline{\mathbb{C}}$. This means that $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a biholomorphic map. Moreover, φ fulfills $\varphi(0) = 0$, $\varphi'(0) = 1$, and $\varphi(\infty) = \infty$, because α_N, α_M, and η_{NM} do so. We conclude with Lemma 3.2.11 that φ is the identity map, which is equivalent to $\alpha_M \equiv \alpha_N \circ \eta_{NM}$.
4. Step: Defining c and g:
The function $\alpha_N : \overline{\mathbb{C}}\backslash\mathbb{D} \to \overline{\mathbb{C}}$ is univalent on the interior $\overline{\mathbb{C}}\backslash\overline{\mathbb{D}}$ and $\mathcal{C}^N$ on the boundary $\partial\mathbb{D}$. Taking $\eta_{NM}|_{\overline{\mathbb{C}}\backslash\mathbb{D}} \equiv \mathsf{id}$ into account the third step yields

$$\alpha_N|_{\overline{\mathbb{C}}\backslash\mathbb{D}} \equiv \alpha_M|_{\overline{\mathbb{C}}\backslash\mathbb{D}}$$

for all $N, M \in \mathbb{N}$ and we can define

$$\alpha : \overline{\mathbb{C}}\backslash\mathbb{D} \to \overline{\mathbb{C}}, \quad z \mapsto \alpha_N(z)$$

independently of $N \in \mathbb{N}$. This function is univalent on the interior $\overline{\mathbb{C}}\backslash\overline{\mathbb{D}}$ and smooth up to the boundary $\partial\mathbb{D}$.

Since $\alpha_N : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a $\mathcal{C}^1$-isomorphism with $\alpha_N(0) = 0$, we have $\alpha(z) = \alpha_N(z) \neq 0$ for all $z \in \overline{\mathbb{C}} \backslash \mathbb{D}$. Since $\infty \in \overline{\mathbb{C}}$ is a fixed point of α, the expression $c := \alpha'(\infty)$ is well-defined. Moreover, we have $c \in \mathbb{C}^\times$ due the the fact that α is $\mathcal{C}^1$-isomorphism. Now we can define the function

$$H : \overline{\mathbb{D}} \to \overline{\mathbb{C}}, \quad z \to \frac{c}{\alpha(\frac{1}{z})},$$

which is smooth and injective on $\overline{\mathbb{D}}$ and univalent on $\mathbb{D}$. Moreover, it satisfies $H(0) = 0$, $H'(0) = 1$, and $H'(z) \neq 0$ for all $z \in \overline{\mathbb{D}}$. Therefore, Lemma 2.4.7 yields $h \in \mathtt{V}^+$ for the function

$$h : \mathbb{S}^1 \to \mathbb{C}, \quad t \mapsto H(e^{it}),$$

and by Lemma 2.4.10 we have $g \in \mathtt{V}^-$ for the function

$$g : \mathbb{S}^1 \to \mathbb{C} \quad t \mapsto \frac{1}{h(-t)}.$$

5. Step: Defining f:
For each $N \in \mathbb{N}$ the function

$$\alpha_N \circ \Gamma_N^{-1} : \overline{\mathbb{D}} \to \overline{\mathbb{C}}$$

is N-times continuously differentiable, injective, and univalent on the interior of $\mathbb{D}$. Taking $\eta_{NM}|_{\mathbb{D}} = \Gamma_N^{-1} \circ \Gamma_M$ into account, the third step implies

$$\alpha_N \circ \Gamma_N^{-1}|_{\mathbb{D}} \equiv \alpha_M \circ \Gamma_M^{-1}|_{\mathbb{D}}$$

for all $N, M \in \mathbb{N}$. Therefore,

$$\beta : \overline{\mathbb{D}} \to \overline{\mathbb{C}}, \quad z \to \alpha_N \circ \Gamma_N^{-1}(z)$$

is independent of N, injective, univalent on the interior $\mathbb{D}$, and smooth up to the boundary $\partial\mathbb{D}$. Since the function α_N is a $\mathcal{C}^1$-isomorphism and $\partial_z \Gamma_N(e^{it}) \neq 0$, we obtain that $\beta'(z) \neq 0$ for all $z \in \overline{\mathbb{D}}$. Moreover,

$$\beta(0) = \alpha_N(\Gamma_N^{-1}(0)) = 0 \qquad \text{and} \qquad \beta'(0) = \alpha_N'(\Gamma_N^{-1}(0)) \frac{1}{\Gamma_N'(0)} = 1.$$

We define the function $f : \mathbb{S}^1 \to \mathbb{C},\ t \mapsto \beta(e^{it}),$ and Lemma 2.4.7 yields $f \in \mathtt{V}^+$.

6. Step: $f(\gamma(t)) = c \cdot g(t)$:
We have

$$\beta(\tilde{\gamma}(z)) = \alpha(z)$$

for all $z \in \partial\mathbb{D}$, and therefore

$$\begin{aligned} f(\gamma(t)) &= \beta(e^{i\gamma(t)}) = \beta(\tilde{\gamma}(e^{it})) = \alpha(e^{it}) \\ &= c \cdot [\frac{c}{\alpha(\frac{1}{e^{-it}})}]^{-1} = c \cdot \frac{1}{H(e^{-it})} = c \cdot \frac{1}{h(-t)} \\ &= c \cdot g(t). \end{aligned}$$

7. Step: Uniqueness:
It remains to show that the functions f and g together with the complex number c of the previous steps are the only solution. Let $\tilde{f} \in \mathtt{V}^+$, $\tilde{g} \in \mathtt{V}^-$ and $\tilde{c} \in \mathbb{C}^\times$ be any solution such that $\tilde{f} \circ \gamma = \tilde{c} \cdot \tilde{g}$. By Lemma 2.3.8, there exist two smooth functions $F, \tilde{F} : \overline{\mathbb{D}} \to \mathbb{C}$ which are holomorphic on the interior $\mathbb{D}$ and fulfill

$$f(t) = F(e^{it}) \qquad \text{and} \qquad \tilde{f}(t) = \tilde{F}(e^{it})$$

for all $t \in \mathbb{S}^1$. Furthermore, $F(\overline{\mathbb{D}}) = \overline{\mathtt{D}}_f$ and $\tilde{F}(\overline{\mathbb{D}}) = \overline{\mathtt{D}}_{\tilde{f}}$. In a similar way, we see that there exist two smooth functions $G, \tilde{G} : \overline{\mathbb{C}} \backslash \mathbb{D} \to \overline{\mathbb{C}}$ which are holomorphic on the interior $\overline{\mathbb{C}} \backslash \overline{\mathbb{D}}$ and fulfill

$$g(t) = c \cdot G(e^{it}) \qquad \text{and} \qquad \tilde{g}(t) = \tilde{c} \cdot \tilde{G}(e^{it})$$

for all $t \in \mathbb{S}^1$. We also have

$$G(\overline{\mathbb{C}} \backslash \mathbb{D}) = \overline{\mathbb{C}} \backslash \mathtt{D}_f \qquad \text{and} \qquad \tilde{G}(\overline{\mathbb{C}} \backslash \mathbb{D}) = \overline{\mathbb{C}} \backslash \mathtt{D}_{\tilde{f}}.$$

Now we are able to define two continuous functions by

$$\begin{aligned} a := \tilde{F} \circ F^{-1} &: \overline{\mathtt{D}}_f \to \overline{\mathtt{D}}_{\tilde{f}}, \quad \text{and} \\ b := \tilde{G} \circ G^{-1} &: \overline{\mathbb{C}} \backslash \mathtt{D}_f \to \overline{\mathbb{C}} \backslash \mathtt{D}_{\tilde{f}}. \end{aligned}$$

Due to

$$\begin{aligned} a|_{\partial \mathtt{D}_f} &\equiv \tilde{F} \circ F^{-1}|_{\partial \mathtt{D}_f} \equiv \tilde{f} \circ f^{-1}|_{\partial \mathtt{D}_f} \equiv (\tilde{f} \circ \gamma) \circ (f \circ \gamma)^{-1}|_{\partial \mathtt{D}_f} \\ &\equiv (\tilde{c} \cdot \tilde{g}) \circ (c \cdot g)^{-1}|_{\partial \mathtt{D}_f} \equiv \tilde{G} \circ G^{-1}|_{\partial \mathtt{D}_f} \equiv b|_{\partial \mathtt{D}_f}, \end{aligned}$$

we can define a continuous function on the whole Riemann sphere by

$$E : \overline{\mathbb{C}} \to \overline{\mathbb{C}}, \quad z \to \begin{cases} a(z) & | \quad z \in \overline{\mathtt{D}}_f \\ b(z) & | \quad z \in \overline{\mathbb{C}} \backslash \mathtt{D}_f \end{cases} .$$

The restriction of E to $\overline{\mathbb{C}} \backslash \partial \mathtt{D}_f$ is holomorphic. This implies that E is holomorphic on $\overline{\mathbb{C}}$, because

$$\oint_C E(z)\, dz = 0$$

is satisfied for every closed curve in $\mathbb{C}$. Since $F(0) = 0$, $\tilde{F}(0) = 0$, $F'(0) = 1$, $\tilde{F}'(0) = 1$, $G(\infty) = \infty$, and $\tilde{G}(\infty) = \infty$, we have

$$E(0) = 0, E'(0) = 1, \text{ and } \quad E(\infty) = \infty.$$

With Lemma 3.2.11 we conclude that E is the indentity map on the Riemann sphere $\overline{\mathbb{C}}$. This has the two consequences $\tilde{F} \equiv F$ and $\tilde{G} \equiv G$, so that we obtain $\tilde{f} \equiv f$ and $\tilde{c} \cdot \tilde{g} \equiv c \cdot g$. In other words, we have shown the uniqueness of the Birkhoff decomposition. $\square$

Chapter 4

Bijectivity of the Composition Map

Recall, that $\mathtt{V}^E \subseteq \mathtt{V}$ was defined in 2.4.3 to be the subset of functions of the form $f(t) = e^{it}(r + a_1 e^{it} + \cdots)$ with Fourier coefficients $r \in \mathbb{R}^+$ and $a_1, a_2, \ldots \in \mathbb{C}$. In this chapter, we will show that the composition map

$$\mathtt{C} : \mathtt{V}^E \times \mathsf{Diff}^+\mathbb{S}^1 \to \mathtt{V}, \quad (f, \gamma) \mapsto f \circ \gamma$$

is bijective (Theorem 4.2.1).

4.1 Preparation

Theorem 4.1.1 (Riemann Mapping Theorem)**.** *Let $G \subsetneq \mathbb{C}$ be a simply connected domain other than the plane itself. Then there exists a biholomorphic map $\varphi : \mathbb{D} \to G$.*

Proof. For a proof see [28] on page 274. □

Lemma 4.1.2 (Biholomorphic maps of $\mathbb{D}$)**.** *All biholomorphic maps on $\mathbb{D}$ are of the form*

$$\mathbb{D} \to \mathbb{D}, \quad z \mapsto e^{i\theta} \frac{z - b}{1 - \bar{b}z}$$

with parameters $\theta \in \mathbb{R}$ and $b \in \mathbb{D}$.

Proof. See Theorem 1.7 on page 6 of [20]. □

Lemma 4.1.3. *Let $G \subsetneq \mathbb{C}$ be a simply connected domain other than the plane itself with $0 \in G$. Then there exists exactly one biholomorphic map $F : \mathbb{D} \to G$ such that*
(i) $F(0) = 0$ *and*
(ii) $F'(0) \in \mathbb{R}^+$.

Proof. 1. Step: Construction of F:
We start with an arbitrary biholomorphic map $\varphi : \mathbb{D} \to G$, which exists by the Riemann Mapping Theorem 4.1.1. The function

$$\rho_1(z) := \frac{z + \varphi^{-1}(0)}{1 + \overline{\varphi^{-1}(0)} \cdot z}$$

is biholomorphic on $\mathbb{D}$, such that $\varphi \circ \rho_1$ is biholomorphic and satisfies condition (i). Moreover, the function

$$F(z) := (\varphi \circ \rho_1)(e^{i\theta} \cdot z)$$

with the complex phase

$$e^{i\theta} := \frac{|(\varphi \circ \rho_1)'(0)|}{(\varphi \circ \rho_1)'(0)}$$

satisfies (i) and (ii).
2. Step: Uniqueness of F:
Assume the two biholomorphic maps $F_1, F_2 : \mathbb{D} \to G$ satisfy the conditions (i) and (ii). By Lemma 4.1.2, the composition

$$\varphi : \mathbb{D} \to \mathbb{D}, \quad z \mapsto F_1^{-1}(F_2(z))$$

is of the form

$$\varphi(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}$$

and we have

$$\varphi(0) = e^{i\theta} a \qquad \text{and} \qquad \varphi'(0) = e^{i\theta}(1 - a\bar{a}). \qquad (\dagger)$$

Moreover,

$$\varphi(0) = 0 \qquad \text{and} \qquad \varphi'(0) \in \mathbb{R}^+ \qquad (\dagger\dagger)$$

and the combination of $(\dagger)$ with $(\dagger\dagger)$ yields

$$0 = e^{i\theta} a \qquad \text{and} \qquad e^{i\theta}(1 - a\bar{a}) \in \mathbb{R}.$$

This implies $a = 0$ and $e^{i\theta} = 1$. Hence, φ is the identity map and $F_1 \equiv F_2$.
□

Definition 4.1.4. We call the positive real number $F'(0)$ in the lemma above the *conformal radius* of the domain G.

Theorem 4.1.5. *Let $D \subsetneq \mathbb{C}$ be a bounded, simply connected domain which is bounded by a smooth Jordan curve. If $F : \mathbb{D} \to D$ is any biholomorphic mapping, then F and all its derivatives have continuous extensions to the closure of D. Furthermore, F^{-1} and all its derivatives have continuous extensions to the closure of D.*

Proof. This is the main result of [3]. $\square$

Corollary 4.1.6. *If $f \in \mathtt{V}$ and $F : \mathbb{D} \to \mathtt{D}_f$ is a biholomorphic map, then there exists a continuous extension $\tilde{F} : \overline{\mathbb{D}} \to \overline{\mathtt{D}}_f$ whose restriction to the boundary*

$$\tilde{F} : \partial\mathbb{D} \to \partial\mathtt{D}_f$$

is a diffeomorphism.

4.2 Theorem and Example

Theorem 4.2.1. *The composition map*

$$\mathtt{C} : \mathtt{V}^E \times \mathsf{Diff}^+\mathbb{S}^1 \to \mathtt{V}, \quad (f, \gamma) \mapsto f \circ \gamma$$

is bijective.

Proof. 1. Step: Surjectivity:
Let $\eta \in \mathtt{V}$. We will show there exists a function $f \in \mathtt{V}^E$ and a diffeomorphism $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$ such that $\eta = f \circ \gamma$. According to Lemma 4.1.3, there exists exactly one biholomorphic map

$$F : \mathbb{D} \to \mathtt{D}_\eta$$

with $F(0) = 0$ and $F'(0) \in \mathbb{R}^+$. By Corollary 4.1.6, there exists a continuous extension $\tilde{F} : \overline{\mathbb{D}} \to \overline{\mathtt{D}}_f$ such that the restriction to the boundary is a diffeomorphism, i.e.,

$$f : \mathbb{S}^1 \to \mathbb{C}, \quad t \mapsto \tilde{F}(e^{it})$$

is a diffeomorphic embedding. Since f and η map $\mathbb{S}^1$ diffeomorphically to $\partial\mathtt{D}_f$, there exists a diffeomorphism $\gamma : \mathbb{S}^1 \to \mathbb{S}^1$ such that

$$\eta = f \circ \gamma.$$

This diffeomorphism γ is an element of $\mathsf{Diff}^+\mathbb{S}^1$, because the winding number of f and η are both equal to 1. The Taylor series of the function $F : \mathbb{D} \to \mathbb{C}$ is

$$F(z) = z(s + a_1 z + a_2 z^2 + \cdots)$$

with $s \in \mathbb{R}^+$ and $a_1, a_2, \ldots \in \mathbb{C}$, and therefore the Fourier series of f is

$$f(t) = e^{it}(s + a_1 e^{it} + a_2 e^{2it} + \cdots),$$

because F extends continuously to the function f in the boundary $\partial\mathbb{D}$. Finally, we conclude that $f \in \mathtt{V}^E$.
2. Step: Injectivity:
Given $f_1, f_2 \in \mathtt{V}^E$ and $\gamma_1, \gamma_2 \in \mathsf{Diff}^+\mathbb{S}^1$ and assume

$$f_1 \circ \gamma_1 = f_2 \circ \gamma_2.$$

By Lemma 2.4.6, there exist two biholomorphic maps

$$F_1, F_2 : \mathbb{D} \to \mathtt{D}_\eta$$

whose Taylor coefficients are the Fourier coefficients of f_1 respectively f_2. Since $F_k(0) = 0$ and $F_k'(0) \in \mathbb{R}^+$ for $k = 1, 2$, Lemma 4.1.3 yields $F_1 = F_2$. Hence, $f_1 = f_2$ and $\gamma_1 = \gamma_2$ and the injectivity of the composition map is shown. □

Definition 4.2.2. According to Theorem 4.2.1, we write

$$\hat{\Lambda} : \mathtt{V} \to \mathtt{V}^E, \quad f \mapsto \hat{\Lambda}(f) = \pi_1 \circ \mathtt{C}^{-1}$$

for the left component of the inverse function and

$$\hat{\Gamma} : \mathtt{V} \to \mathsf{Diff}^+\mathbb{S}^1, \quad f \mapsto \hat{\Gamma}(f) = \pi_2 \circ \mathtt{C}^{-1}$$

for the right component, where π_1 denotes the projections onto the left respective and π_2 onto the right component.

Corollary 4.2.3. *If $f \in \mathtt{V}$, then*

(i) $f = \hat{\Lambda}(f) \circ \hat{\Gamma}(f)$, *and*

(ii) $f'(t) = \hat{\Lambda}(f)'(\hat{\Gamma}(f)(t)) \cdot \hat{\Gamma}(f)'(t)$.

(iii) $\hat{\Lambda}|_{\mathtt{V}^E} \equiv \mathsf{id}|_{\mathtt{V}^E}$.

Example 4.2.4. Here is an example of the preceding construction. Let $n \in \mathbb{N}$, $\lambda \in \mathbb{R}$, and

$$f(t) \;=\; e^{it}(1+e^{-int}\,\mathsf{tanh}\,\lambda)^{\frac{1}{n}},$$

then

$$\begin{aligned}\hat{\Gamma}(f)(t) &= \frac{2}{n}\mathsf{arctan}(e^{-2\lambda}\tan\frac{nt}{2}), \text{ and}\\ \hat{\Lambda}(f)(\tau) &= e^{i\tau}(1-\mathsf{tanh}^2\lambda)^{\frac{1}{n}}(1-e^{in\tau}\mathsf{tanh}\lambda)^{-\frac{1}{n}}.\end{aligned}$$

Proof.

$$\begin{aligned}\hat{\Gamma}(f)(t) = &\; \frac{2}{n}\mathsf{arctan}(e^{-2\lambda}\mathsf{tan}\frac{nt}{2})\\ = &\; \frac{1}{in}\mathsf{log}\frac{1+e^{-2\lambda}i\,\mathsf{tan}\frac{nt}{2}}{1-e^{-2\lambda}i\,\mathsf{tan}\frac{nt}{2}}\\ = &\; \frac{1}{in}\mathsf{log}\frac{e^{\lambda}(e^{in\frac{t}{2}}+e^{-in\frac{t}{2}})+e^{-\lambda}(e^{in\frac{t}{2}}-e^{-in\frac{t}{2}})}{e^{\lambda}(e^{in\frac{t}{2}}+e^{-in\frac{t}{2}})-e^{-\lambda}(e^{in\frac{t}{2}}-e^{-in\frac{t}{2}})}\\ = &\; \frac{1}{in}\mathsf{log}\frac{e^{in\frac{t}{2}}\mathsf{cosh}\lambda+e^{-in\frac{t}{2}}\mathsf{sinh}\lambda}{e^{-in\frac{t}{2}}\mathsf{cosh}\lambda+e^{in\frac{t}{2}}\mathsf{sinh}\lambda}\\ = &\; \frac{1}{in}\mathsf{log}\frac{e^{int}+\mathsf{tanh}\lambda}{1+e^{int}\mathsf{tanh}\lambda},\end{aligned}$$

implies

$$e^{-in\hat{\Gamma}(f)(t)} = \frac{1+e^{int}\mathsf{tanh}\lambda}{e^{int}+\mathsf{tanh}\lambda} = \frac{e^{int}\mathsf{sinh}\,\lambda+\mathsf{cosh}\,\lambda}{e^{int}\mathsf{cosh}\,\lambda+\mathsf{sinh}\,\lambda}.$$

Moreover,

$$\hat{\Lambda}(f)(\tau) = (1-\mathsf{tanh}^2\lambda)^{\frac{1}{n}}(e^{-in\tau}-\mathsf{tanh}\lambda)^{-\frac{1}{n}},$$

and finally

$$\begin{aligned}&(\hat{\Lambda}(f)\circ\hat{\Gamma}(f))(t)\\ = &\; (1-\mathsf{tanh}^2\lambda)^{\frac{1}{n}}\left(\frac{1+e^{int}\mathsf{tanh}\lambda}{e^{int}+\mathsf{tanh}\lambda}-\mathsf{tanh}\lambda\right)^{-\frac{1}{n}}\\ = &\; (1-\mathsf{tanh}^2\lambda)^{\frac{1}{n}}\left(\frac{1+e^{int}\mathsf{tanh}\lambda}{e^{int}+\mathsf{tanh}\lambda}-\mathsf{tanh}\lambda\frac{(e^{int}+\mathsf{tanh}\lambda)}{(e^{int}+\mathsf{tanh}\lambda)}\right)^{-\frac{1}{n}}\\ = &\; (e^{int}+\mathsf{tanh}\lambda)^{\frac{1}{n}}\\ = &\; e^{it}(1+e^{-int}\,\mathsf{tanh}\,\lambda)^{\frac{1}{n}}\\ = &\; f(t).\end{aligned}$$

$\square$

Chapter 5

Tameness

In later chapters, we need an inverse function theorem in the framework of Fréchet spaces. Although the well-known Inverse-Function-Theorem for $\mathbb{R}^n$ carries over without problems to Banach spaces, one needs additional technical requirements for Fréchet spaces. The main such requirements are the tameness of spaces (5.1.5) and the tameness of maps (5.2.4). In this chapter, we start with the definition and basic properties. The corresponding theorem is called the Inverse-Function-Theorem of Nash and Moser or the Nash-Moser Theorem. We will formulate it at the end of Section 2. One of the additional requirements for the Nash-Moser Theorem is that the derivative is invertible, not only at a single point, but on an open neighborhood. We spend most of Chapters 7 and 8 to verify this stronger requirement. In Section 3 and 4, we compute some derivatives of functions between function spaces.

5.1 Linear Tame Spaces and Maps

In this section, we develop the concept of tameness for linear spaces. We use the definitions from the article of Hamilton [13].

Definition 5.1.1 (Graded Fréchet space)**.** A *grading* on a Fréchet space V is an increasing series of semi-norms

$$\|f\|_0 \leq \|f\|_1 \leq \|f\|_2 \leq \cdots$$

which defines the Fréchet topology. A *graded Fréchet space* is one with a choice of a grading.

Definition 5.1.2 (Linear-tame map)**.** We say, that a linear function

$$L : F \to G$$

between graded Fréchet spaces is a *linear-tame map* if there exist two natural numbers $r, b \in \mathbb{N}_0$ and a sequence of constants $(C_n)_{n\in\mathbb{N}}$ such that

$$\|Lf\|_n \leq C_n \|f\|_{n+r}$$

for each $n \geq b$.

Remark: A linear-tame map is automatically continuous with respect to the Fréchet topology. Moreover, $\mathrm{Hom}(F, G)_{tame}$ is a vector space.

Definition 5.1.3 (Tame direct summand)**.** Let F and G be graded Fréchet spaces. We say that F is a *tame direct summand* of G if we can find linear-tame maps $L : F \to G$ and $M : G \to F$ such that the composition

$$M \circ L : F \to F$$

is the identity on F.

Example 5.1.4 (Graded Fréchet space)**.** Let $\Sigma(B)$ denote the space of all sequences $(f_k)_{k\in\mathbb{N}}$ of elements in a Banach space B such that

$$\|(f_k)_{k\in\mathbb{N}}\|_n = \sum_{k=0}^{\infty} e^{nk} \|f_k\|_B < \infty$$

for all $n \geq 0$. Then $\Sigma(B)$ is a graded Fréchet space with the norms above.

Definition 5.1.5 (Tame space)**.** We say a graded Fréchet space is a *tame space* if it is a tame direct summand of a space $\Sigma(B)$ of exponentially decreasing sequences in some Banach space B.

Remark: Every tame space is a Fréchet space.

Definition 5.1.6 (Cartesian product of graded Fréchet spaces)**.** Let V and W be two graded Fréchet spaces. The grading on the cartesian product $V \times W$ is defined by

$$\|(f, g)\|_n := \sum_{j=0}^{n} \|f\|_j^V + \|g\|_{n-j}^W$$

where $(\| \cdot \|_n^V)_{n\in\mathbb{N}_0}$ is the grading of V and $(\| \cdot \|_n^W)_{n\in\mathbb{N}_0}$ the grading of W.

Lemma 5.1.7. *A cartesian product of two tame spaces is tame.*

Proof. See [13] Lemma 1.3.4 on page 136. □

Definition 5.1.8 (Gradings)**.** We define gradings on the vector spaces $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ and $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$ by

$$\|f\|_n := \sum_{k=0}^{n} \mathsf{sup}_{t\in\mathbb{S}^1} |f^{(k)}(t)|,$$

where n runs over the natural numbers $\mathbb{N}_0$. Moreover, on $\mathcal{C}^\infty(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{C})$ we define the grading

$$\|q\|_n := \sum_{k=0}^{n} \sum_{j=0}^{k} \mathsf{sup}_{\theta,\tau\in\mathbb{S}^1} \left| \left(\frac{\partial}{\partial\theta}\right)^{k-j} \left(\frac{\partial}{\partial\tau}\right)^{j} q(\theta, \tau) \right|$$

for $q \in \mathcal{C}^\infty(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{C})$ and $n \in \mathbb{N}_0$.

Remark: These gradings define the usual Fréchet topologies. Moreover, we have
$\|f\|_{n+k+1} = \|f^{(k+1)}\|_n + \|f\|_k$ for $f \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$.

Lemma 5.1.9. *The spaces $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$, $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$ and $\mathcal{C}^\infty(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{C})$ endowed with the gradings of Definition 5.1.8 are tame spaces.*

Proof. Since $\mathbb{S}^1$ and $\mathbb{S}^1 \times \mathbb{S}^1$ are compact manifolds, we can apply Theorem 1.3.6. of [13] on page 137. A more general version of this theorem can be found on page 146, Corollary 2.3.2. □

5.2 Smooth-Tame Maps

In this section, we extend the concept of tameness to arbitrary smooth maps. At the end of this section, we formulate the Nash-Moser Theorem. Mainly, we refer to Hamilton [13] for the proofs.

Definition 5.2.1 (Set of all smooth functions)**.** Let A and B be two finite-dimensional manifolds. We write $\mathcal{C}^\infty(A, B)$ for the *set of all smooth functions* from A to B.

Definition 5.2.2 (Left- and right composition map)**.** Consider two finite-dimensional manifolds X and Z and a smooth map $\rho : A \to B$ between two finite-dimensional manifolds A and B, we define

(i) the *left composition* by
$\mathsf{L}_\rho : \mathcal{C}^\infty(X, A) \to \mathcal{C}^\infty(X, B), \quad \gamma \mapsto \mathsf{L}_\rho(\gamma) := \rho \circ \gamma$

(ii) and the *right composition* by
$\mathsf{R}_\rho : \mathcal{C}^\infty(B, Z) \to \mathcal{C}^\infty(A, Z), \quad f \mapsto \mathsf{R}_\rho(f) := f \circ \rho.$

Definition 5.2.3 (Tame map)**.** Let F and G be graded Fréchet spaces and $U \subseteq F$ an open subset. We say that $P : U \to G$ is a *tame map* if for every $f \in U$ there exists an open neighborhood U_f, two natural numbers $r, b \in \mathbb{N}_0$, and a series of positive real constants $(C_n)_{n \in \mathbb{N}_0}$ such that

$$\|P(g)\|_n \leq C_n(1 + \|g\|_{n+r})$$

for all $g \in U_f$ and all $n \geq b$.

Definition 5.2.4 (Smooth-tame map)**.** Let F and G be graded Fréchet spaces, $U \subseteq F$ an open subset. We say $P : U \to G$ is a *smooth-tame map* if P is smooth and all its derivatives $d^k P : U \times F^k \to G$ are tame.

Definition 5.2.5 (Tame diffeomorphism)**.** Let F and G be graded Fréchet spaces, and let $U \subseteq F$, $W \subseteq G$ be two open subsets. We say the map

$$P : U \to W$$

is a *tame diffeomorphism* if P is smooth-tame, bijective, and its inverse P^{-1} is also smooth-tame.

Lemma 5.2.6. *A map is linear-tame if and only if it is linear and tame.*

Proof. This is Theorem 2.1.5 in [13] on page 141. □

Lemma 5.2.7. *A linear-tame map is a smooth-tame map.*

Proof. All derivatives of a linear-tame map are also linear-tame maps. □

Lemma 5.2.8. *(i) The composition of tame maps is a tame map.*

(ii) The composition of linear-tame maps is a linear-tame map.

(iii) The composition of smooth-tame maps is a smooth-tame map.

Proof. (i): This is Theorem 2.1.6 in [13] on page 142.
(ii): The composition of linear maps is linear again. The same is true for tame maps. By Lemma 5.2.6, a map is linear-tame if and only if it tame and linear.
(iii): By Definition 5.2.4, a smooth-tame map is a map such that all derivatives are tame maps. Let $f : L \to M$ and $g : M \to N$ be two smooth-tame maps between open subsets of tame spaces. Then the n-th derivatives

$$\underbrace{d(d(\cdots(d}_{n-\text{times}} f)\cdots)) : \underbrace{T(T(\cdots(}_{n-\text{times}} L)\cdots)) \to T(T(\cdots(M)\cdots))$$

and

$$d(d(\cdots(dg)\cdots)) : T(T(\cdots(M)\cdots)) \to T(T(\cdots(N)\cdots))$$

are tame maps. The n-th derivatives of the composition $g \circ f$ is the composition of both n-th derivatives, i.e.,

$$d(d(\cdots(d(g \circ c))\cdots)) = d(d(\cdots(dg)\cdots)) \;\circ\; d(d(\cdots(df)\cdots)),$$

which is tame by (i). Since this is true for all $n \in \mathbb{N}$, the assertion is proved due to Definition 5.2.4. $\square$

Lemma 5.2.9. *Let $U \subseteq \mathbb{C}$ be an open subset, then $\mathcal{C}^\infty(\mathbb{S}^1, U) \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is an open subset with respect to the Fréchet topology.*

Proof. Let $f \in \mathcal{C}^\infty(\mathbb{S}^1, U)$. Then the image $f(\mathbb{S}^1)$ is compact, thus we can find an $\epsilon > 0$ such that the ϵ-tube around $f(\mathbb{S}^1)$ lies completely in U. Hence, the ϵ-ball in $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ around f with respect to the semi-norm $\|\cdot\|_0$ lies in $\mathcal{C}^\infty(\mathbb{S}^1, U)$. This is a neighborhood of f in $\mathcal{C}^\infty(\mathbb{S}^1, U)$. $\square$

Lemma 5.2.10. *(i) The derivative map*

$$\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \lambda \mapsto \lambda',$$

(ii) and the integration map

$$I : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \lambda \mapsto I.\lambda$$

with

$$(I.\lambda)(\theta) = \int_0^\theta \lambda(t)dt - \int_0^{2\pi} \int_0^\xi \lambda(t)dt\, d\xi$$

are linear-tame, and hence smooth-tame as well as continuous.

Proof. (i): We have $\|f\|_{n+1} = \|f'\|_n + \|f\|_0$ and thus $\|f'\|_n \leq \|f\|_{n+1}$. (ii): We have $(I.\lambda)'(t) = \lambda(t)$, which implies $\|I.\lambda\|_{n+1} = \|\lambda\|_n + \|I.\lambda\|_0$ for every $n \in \mathbb{N}_0$. □

Lemma 5.2.11. *Consider a finite-dimensional manifolds Y and a smooth map $\beta : \mathbb{S}^1 \to \mathbb{S}^1$. Then the right composition*

$$\mathsf{R}_\beta : \mathcal{C}^\infty(\mathbb{S}^1, Y) \to \mathcal{C}^\infty(\mathbb{S}^1, Y), \quad \lambda \mapsto \lambda \circ \beta$$

is a linear-tame map, and hence a smooth-tame map.

Proof. This is a consequence of Theorem 2.3.3 on page 147 in [13] if we take into account that a constant map is always tame. □

Lemma 5.2.12. *Let Z be a finite-dimensional manifold. Then the composition map*

$$\mathtt{C} : \mathcal{C}^\infty(\mathbb{S}^1, Z) \times \mathsf{Diff}^+\mathbb{S}^1 \to \mathcal{C}^\infty(\mathbb{S}^1, Z), \quad (f, \gamma) \mapsto f \circ \gamma$$

is a smooth-tame map.

Proof. This is the statement of Theorem 2.3.3 on page 147 in [13] after the substitution $X = \mathbb{S}^1$ and $Y = \mathbb{S}^1$. □

Lemma 5.2.13. *The group of diffeomorphisms $\mathsf{Diff}^+\mathbb{S}^1$ is a smooth-tame Lie group. In particular, we have that*

(i) *the composition map*

$$\mathtt{C} : \mathsf{Diff}^+\mathbb{S}^1 \times \mathsf{Diff}^+\mathbb{S}^1 \to \mathsf{Diff}^+\mathbb{S}^1$$

is smooth-tame, and

(ii) *the inversion map*

$$\iota : \mathsf{Diff}^+\mathbb{S}^1 \to \mathsf{Diff}^+\mathbb{S}^1$$

is smooth-tame.

Proof. This is the statement of Theorem 2.3.5 on page 148 in [13] for the special case $X = \mathbb{S}^1$. □

Definition 5.2.14 (Inverse with respect to the right argument)**.** Let V and W be two vector spaces, $U \subseteq V$ an open subset and

$$F : U \times V \to W, \quad (u, v) \mapsto F(u, v)$$

a function. We say

$$F^{\#} : U \times W \to V, \quad (v, w) \mapsto F^{\#}(u, w)$$

is an *inverse with respect to the right argument* if

$$F^{\#}(u, F(u, v)\,) = v$$

holds for all $v \in V$, and

$$F(u, F^{\#}(u, w)\,) = w$$

holds for all $w \in W$.

Lemma 5.2.15 (Theorem of Nash and Moser)**.** *Let V and W be two tame spaces, $U_V \subseteq V$ an open subset, and $F : U_V \to W$ a smooth-tame map which is injective. Moreover, let*

$$dF : U_V \times V \to W$$

be its derivative. If dF admits an inverse with respect to the right argument

$$dF^{\#} : U_V \times W \to V$$

which is smooth-tame, then $F : U_V \to F(U_V) \subseteq W$ is a tame diffeomorphism, i.e., $F^{\#} : F(U_V) \to U_V$ is also smooth-tame. In particular $F(U_V) \subseteq W$ is open.

Proof. For a proof see [13] on page 171. $\square$

5.3 Tameness of the Left Composition

In this section, we show (5.3.5) that the left composition

$$\mathsf{L}_\alpha : \mathcal{C}^\infty(\mathbb{S}^1, U) \to \mathcal{C}^\infty(\mathbb{S}^1, V), \quad f \mapsto \alpha \circ f$$

is a smooth-tame map for two open subsets $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^l$ and a smooth map $\alpha : U \to V$. This proof is mainly a modification of the proof of Theorem 2.2.5 on page 145 in [13].

Lemma 5.3.1. *For a smooth 2π-periodic function $f \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ the inequality*

$$|\mathsf{sup}_{t\in\mathbb{S}^1}|f'(t)|^2 \leq 4\mathsf{sup}_{t\in\mathbb{S}^1}|f(t)| \cdot \mathsf{sup}_{t\in\mathbb{S}^1}|f''(t)|$$

holds.

Proof. Choose $t_0 \in \mathbb{R}$ such that

$$f'(t_0) = \mathsf{sup}_{t\in\mathbb{S}^1}|f'(t)|$$

holds, and let

$$t_1 := t_0 + \frac{\mathsf{sup}_{t\in\mathbb{S}^1}|f'(t)|}{\mathsf{sup}_{t\in\mathbb{S}^1}|f''(t)|}.$$

Elementary calculus provides us with

$$f(t_1) = f(t_0) + (t_1 - t_0)f'(t_0) + \int_{t_0}^{t_1}\int_{t_0}^{\tau} f''(\theta)d\theta d\tau,$$

and applying the triangle inequality yields

$$\begin{aligned}
|f(t_1) - f(t_0)| &\geq |(t_1 - t_0)f'(t_0)| - |\int_{t_0}^{t_1}\int_{t_0}^{\tau} f''(\theta)d\theta d\tau| \\
&\geq (t_1 - t_0) \cdot \mathsf{sup}_{t\in\mathbb{S}^1}|f'(t)| - \frac{1}{2}(t_1 - t_0)^2\mathsf{sup}_{t\in\mathbb{S}^1}|f''(t)| \\
&= \frac{1}{2}\frac{(\mathsf{sup}_{t\in\mathbb{S}^1}|f'(t)|)^2}{\mathsf{sup}_{t\in\mathbb{S}^1}|f''(t)|}.
\end{aligned}$$

Since we have

$$2 \cdot \mathsf{sup}_{t\in\mathbb{S}^1}|f(t)| \geq |f(t_1) - f(t_0)|,$$

we conclude

$$4 \cdot \mathsf{sup}_{t\in\mathbb{S}^1}|f(t)| \geq \frac{(\mathsf{sup}_{t\in\mathbb{S}^1}|f'(t)|)^2}{\mathsf{sup}_{t\in\mathbb{S}^1}|f''(t)|}$$

and we are finished. □

Lemma 5.3.2. *Let $n, k \in \mathbb{N}$, then the inequality*

$$\mathsf{sup}_{t\in\mathbb{S}^1}|f^{(k)}(t)| \quad\leq\quad 2^{k(n-k)}\cdot\left(\mathsf{sup}_{t\in\mathbb{S}^1}|f(t)|\right)^{\frac{n-k}{n}}\cdot\left(\mathsf{sup}_{t\in\mathbb{S}^1}|f^{(n)}(t)|\right)^{\frac{k}{n}}$$

holds for every smooth function $f \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$.

Proof. Fix a smooth 2π-periodic function $f \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$. Define the series $(a_n)_{n\in\mathbb{N}_0}$ by

$$a_n := \mathsf{sup}_{t\in\mathbb{S}^1}|f^{(n)}(t)|, \qquad\qquad (\dagger)$$

and Lemma 5.3.1 provides us with

$$a_{n+1}^2 \leq 4\, a_n\, a_{n+2}. \qquad\qquad (\dagger\dagger)$$

Define the function

$$b : \mathbb{N}_0 \to \mathbb{R}, \quad n \mapsto b_n := n^2 \,\mathsf{log}\, 2 + \mathsf{log}\, a_n.$$

The computation

$$\begin{aligned} b_{n+1} &= (n+1)^2 \mathsf{log}\, 2 + \mathsf{log}\, a_{n+1} \\ &\overset{(\dagger\dagger)}{\leq} (n+1)^2 \mathsf{log}\, 2 + \frac{1}{2}[\mathsf{log}\, 4 + \mathsf{log}\, a_n + \mathsf{log}\, a_{n+2}] \\ &= \frac{1}{2}[n^2\, \mathsf{log}\, 2 + (n+2)^2\, \mathsf{log}\, 2 + \mathsf{log}\, a_n + \mathsf{log}\, a_{n+2}] \\ &= \frac{1}{2}[b_n + b_{n+2}] \end{aligned}$$

shows that $b : \mathbb{N}_0 \to \mathbb{R}$ is convex. This implies

$$b_k \leq \frac{n-k}{n} b_0 + \frac{k}{n} b_n$$

for $n, k \in \mathbb{N}$, which is equivalent to

$$k^2 \mathsf{log} 2 + \mathsf{log}\, a_k \leq \frac{n-k}{n} \mathsf{log}\, a_0 + \frac{k}{n}[n^2 \mathsf{log} 2 + \mathsf{log}\, a_n],$$

and hence equivalent to

$$a_k \leq 2^{k(n-k)} a_0^{\frac{n-k}{n}}\, a_n^{\frac{k}{n}}.$$

Substituting $(\dagger)$ back, yields the result. $\square$

Lemma 5.3.3. *Fix two natural numbers $m, l \in \mathbb{N}$. Let $W \subseteq \mathbb{R}^m$ be an bounded open subset. Moreover, let $\alpha : W \to \mathbb{R}^l$ be a smooth map, and assume that α and all its derivatives are bounded. Then there exists a sequence $(C_n)_{n\in\mathbb{N}}$ of positive real numbers such that*

$$\mathsf{sup}_{t\in\mathbb{S}^1}\|\frac{d^n}{dt^n}\alpha(f(t))\|_\infty \le C_n \cdot \mathsf{sup}_{t\in\mathbb{S}^1}\|f^{(n)}(t)\|_\infty$$

holds for all $n \in \mathbb{N}$ and for all smooth functions

$$f : \mathbb{S}^1 \to W, \quad t \mapsto (f_1(t), f_2(t) \cdots , f_m(t)).$$

We write $\|\cdot\|_\infty$ for the maximum norm on $\mathbb{R}^m$ respectively $\mathbb{R}^l$.

Proof. 1. Step:
For the multi-index $k = (k_1, \dots, k_n) \in {\mathbb{N}_0}^n$ we define

$$\begin{aligned} |k| &:= k_1 + \cdots + k_n, \text{ and} \\ \|k\| &:= k_1 + 2 \cdot k_2 + \cdots + n \cdot k_n. \end{aligned}$$

An induction on n yields that the n-th derivative of $\alpha \circ f$ is

$$\begin{aligned} \frac{d^n}{dt^n}\alpha(f(t)) = & \sum_{\|k\|=n} a_k \sum_{j[1]\,,\,j[2]\,,\dots,\,j[|k|]=1}^{m} (\partial_{j[1]}\partial_{j[2]}\cdots\partial_{j[|k|]}\alpha)(f(t)) \cdot \\ & f'_{j[1]}(t) \cdot f'_{j[2]}(t) \cdot \dots \cdot f'_{j[k_1]}(t) \cdot \\ & f''_{j[k_1+1]}(t) \cdot \dots \cdot f''_{j[k_1+k_2]}(t) \cdot \\ & f'''_{j[k_1+k_2+1]}(t) \cdot \dots \cdot f'''_{j[k_1+k_2+k_3]}(t) \cdot \\ & \vdots \\ & f^{(n)}_{j[k_1+\cdots+k_{n-1}+1]}(t) \cdot \dots \cdot f^{(n)}_{j[k_1+\cdots+k_n]}(t), \end{aligned} \tag{$\dagger$}$$

where a_k is an integer, which depends on the multi-index $k = (k_1, \dots, k_n)$. In the sum $\sum_{\|k\|=n}$ the multi-index $k = (k_1, \dots, k_n)$ runs over all combinations with $k_1 + 2 \cdot k_2 + \cdots + n \cdot k_n = n$, and in the sum $\sum_{j[1],j[2],\dots,j[|k|]=1}^{m}$ the indices $j[1]$, $j[2]$, etc., are distinguished indices running from 1 to m.

2. Step:
By Lemma 5.3.2, we have

$$\begin{aligned} f_j^{(l)}(t) &\leq \mathsf{sup}_{t\in\mathbb{S}^1}|f_j^l(t)| \\ &\leq 2^{l(n-l)}(\mathsf{sup}_{t\in\mathbb{S}^1}|f_j(t)|)^{\frac{n-l}{n}}(\mathsf{sup}_{t\in\mathbb{S}^1}|f_j^{(n)}(t)|)^{\frac{l}{n}} \\ &\leq 2^{l(n-l)}(\mathsf{sup}_{t\in\mathbb{S}^1}\|f(t)\|_\infty)^{\frac{n-l}{n}}(\mathsf{sup}_{t\in\mathbb{S}^1}\|f^{(n)}(t)\|_\infty)^{\frac{l}{n}} \\ &\leq 2^{l(n-l)}\rho_W^{\frac{n-l}{n}}(\mathsf{sup}_{t\in\mathbb{S}^1}\|f^{(n)}(t)\|_\infty)^{\frac{l}{n}} \\ &\leq 2^{n^2}\rho_W(\mathsf{sup}_{t\in\mathbb{S}^1}\|f^{(n)}(t)\|_\infty)^{\frac{l}{n}} \end{aligned}$$

for $j = 1, \ldots, m$. In the equation (†) we have k_1 times the first derivative of f as a factor, k_2 times the second derivative, k_l times the l-th derivative, and so on. All together these are $n = k_1 + 2\,k_2 + \cdots + n\,k_n$ factors. Hence, we can estimate

$$\begin{aligned} & f'_{j[1]}(t)\cdot f'_{j[2]}(t)\cdot\ldots\cdot f^{(n)}_{j[|k|}(t) \\ \leq\ & (2^{n^2}\rho_W)^n\cdot(\mathsf{sup}_{t\in\mathbb{S}^1}\|f^{(n)}(t)\|_\infty)^{\frac{k_1+2\cdot k_2+\cdots+n\cdot k_n}{n}} \\ \leq\ & 2^{n^3}\,\rho_W^n\,\cdot\,\mathsf{sup}_{t\in\mathbb{S}^1}\|f^{(n)}(t)\|_\infty. \end{aligned}$$

3. Step:
For all $n \in \mathbb{N}$ there exists a real number $k_n > 0$ such that

$$|(\partial_{j[1]}\partial_{j[2]}\cdots\partial_{j[|k|]}\alpha)(f(t))\,| < k_n,$$

because all derivatives of $\alpha : W \to \mathbb{R}^n$ are bounded. The constant k_n is independent of the choice of f.
4. Step:
The sum $s_n := \sum_{\|k\|=n} a_k$ depends on n, because a_k depends on the multi-index $k = (k_1, \cdots, k_n)$. Furthermore, we have

$$\sum_{j[1]\,,\,j[2]\,,\,\ldots,j[|k|]=1}^{m} 1 \leq m^n.$$

5. Step:
With the previous steps we conclude that

$$C_n = s_n\cdot m^n\cdot k_n\cdot 2^{n^3}\cdot\rho_W^n$$

satisfies

$$\mathsf{sup}_{t\in\mathbb{S}^1}\|\frac{d^n}{dt^n}\alpha(f(t))\|_\infty \leq C_n\cdot\mathsf{sup}_{t\in\mathbb{S}^1}\|f^{(n)}(t)\|_\infty.$$

□

Lemma 5.3.4. *Consider two open subsets* $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^l$ *and a smooth map* $\alpha : U \to V$. *Then the left composition*

$$\mathsf{L}_\alpha : \mathcal{C}^\infty(\mathbb{S}^1, U) \to \mathcal{C}^\infty(\mathbb{S}^1, V), \quad f \mapsto \alpha \circ f$$

is a tame map.

Proof. Let $g \in \mathcal{C}^\infty(\mathbb{S}^1, U)$. Choose an open neighborhood $U_g \subseteq U$ of the image $g(\mathbb{S}^1)$ which is bounded and has the property that all derivatives of α are bounded on U_g. By Lemma 5.2.9, the subset $\mathcal{C}^\infty(\mathbb{S}^1, U_g) \subseteq \mathcal{C}^\infty(\mathbb{S}^1, U)$ is open, and hence it is an open neighborhood of g. By Lemma 5.3.3, there exists a series of positive real constants $(C_n)_{n\in\mathbb{N}}$ such that

$$\mathsf{sup}_{t\in\mathbb{S}^1} \| \frac{d^n}{dt^n} \alpha(f(t)) \|_\infty \leq C_n \cdot \mathsf{sup}_{t\in\mathbb{S}^1} \| f^{(n)}(t) \|_\infty$$

holds for all functions $f \in \mathcal{C}^\infty(\mathbb{S}^1, U_g)$. Here $\| \cdot \|_\infty$ denotes the maximum norm on $\mathbb{R}^n$. The grading on $\mathcal{C}^\infty(\mathbb{S}^1, U)$ as well as on $\mathcal{C}^\infty(\mathbb{S}^1, V)$ is

$$\|f\|_n := \sum_{k=0}^{n} \mathsf{sup}_{t\in\mathbb{S}^1} \| \frac{d^k}{dt^k} f(t) \|_\infty$$

for f in $\mathcal{C}^\infty(\mathbb{S}^1, U)$ respectively $\mathcal{C}^\infty(\mathbb{S}^1, V)$. Let

$$\tilde{C}_n = \sum_{k=0}^{n} C_k.$$

Then we have

$$\begin{aligned}
\|\mathsf{L}_\alpha(f)\|_n &= \sum_{k=0}^{n} \mathsf{sup}_{t\in\mathbb{S}^1} \| \frac{d^k}{dt^k} \alpha(f(t)) \|_\infty \\
&\leq \sum_{k=0}^{n} C_k \cdot \mathsf{sup}_{t\in\mathbb{S}^1} \| f^{(k)}(t) \|_\infty \\
&\leq \tilde{C}_n \sum_{k=0}^{n} \mathsf{sup}_{t\in\mathbb{S}^1} \| f^{(k)}(t) \|_\infty \\
&\leq \tilde{C}_n (1 + \|f\|_n)
\end{aligned}$$

for all f in $\mathcal{C}^\infty(\mathbb{S}^1, U_g)$. Hence, the conditions of Definition 5.2.3 are verified and we conclude $\mathsf{L}_\alpha : \mathcal{C}^\infty(\mathbb{S}^1, U) \to \mathcal{C}^\infty(\mathbb{S}^1, V)$ is tame. □

Lemma 5.3.5. *Consider two open subsets* $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^l$ *and a smooth map* $\alpha : U \to V$. *Then the left composition*

$$\mathsf{L}_\alpha : \mathcal{C}^\infty(\mathbb{S}^1, U) \to \mathcal{C}^\infty(\mathbb{S}^1, V), \quad f \mapsto \alpha \circ f$$

is a smooth-tame map.

Proof. The n-th derivative of $\mathsf{L}_\alpha : \mathcal{C}^\infty(\mathbb{S}^1, U) \to \mathcal{C}^\infty(\mathbb{S}^1, V)$ is

$$d^n\mathsf{L}_\alpha : \mathcal{C}^\infty(\mathbb{S}^1, U) \times (\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}^m))^n \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}^l)$$

with

$$[d^n\mathsf{L}_\alpha(f, \lambda_1, \cdots, \lambda_n)](t) = d^n\alpha(f(t), \lambda_1(t), \cdots, \lambda_n(t)\,)$$

and $d^n\alpha$ is a map

$$d^n\alpha : U \times (\mathbb{R}^m)^n \to \mathbb{R}^l.$$

This can be shown by induction on n with the computation

$$\begin{aligned} & [d^{n+1}\mathsf{L}_\alpha(f, \lambda_1, \ldots, \lambda_n, \lambda_{n+1})](t) \\ := \ & \mathsf{lim}_{\epsilon\to 0}\frac{1}{\epsilon}[d^n\mathsf{L}_\alpha(f + \epsilon\lambda_{n+1}, \lambda_1, \ldots, \lambda_n) - d^n\mathsf{L}_\alpha(f, \lambda_1, \ldots, \lambda_n)](t) \\ = \ & \mathsf{lim}_{\epsilon\to 0}\frac{1}{\epsilon}[d^n\alpha(f(t) + \epsilon\lambda_{n+1}(t), \lambda_1(t), \ldots) - d^n\alpha(f(t), \lambda_1(t), \ldots)] \\ = \ & d^{n+1}\alpha(f(t), \lambda_1(t), \ldots, \lambda_n(t), \lambda_{n+1}(t)). \end{aligned}$$

Since α is smooth, all derivatives $d^n\alpha$ are also smooth. Lemma 5.3.4 yields that the functions $d^n\mathsf{L}_\alpha = \mathsf{L}_{d^n\alpha}$ for $n \in \mathbb{N}_0$ are tame. And by Definition 5.2.4 the function L_α is smooth tame. □

Lemma 5.3.6. *The maps*

(i) $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad (f, g) \mapsto f \cdot g,$

(ii) $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad (f, g) \mapsto f + g,$ *and*

(iii) $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad (f, g) \mapsto \dfrac{f}{g}$

are smooth-tame.

Proof. We can identify $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ with $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C} \times \mathbb{C})$. The multiplication and addition are smooth maps from $\mathbb{C} \times \mathbb{C}$ into $\mathbb{C}$, and we can apply Lemma 5.3.5. The argument for the division is similar. □

Lemma 5.3.7. *The winding number function*

$$\mathtt{w} : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times) \to \mathbb{Z}, \quad f \mapsto \frac{1}{2\pi i}\int_{\mathbb{S}^1} \frac{f'(t)}{f(t)}dt$$

as defined in 2.4.1 is smooth-tame, and hence continuous.

Proof. By Lemma 5.2.10(i) the derivative map is smooth-tame, by Lemma 5.3.6(iii) the division of functions is smooth-tame, and by Lemma 5.2.10(ii) the integration map is smooth-tame. Hence, the winding number function is smooth-tame. □

5.4 Tameness of a Particular Function

In this section we will show (Proposition 5.4.4), that the map

$$P : \mathbb{V} \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad (f, \lambda) \mapsto P(f, \lambda)$$

with

$$[P(f, \lambda)](\tau) := \frac{i}{2\pi} \int_{\mathbb{S}^1} \frac{\lambda(\theta) - \lambda(\tau)}{f(\theta) - f(\tau)} f'(\theta) d\theta$$

is smooth-tame.

Lemma 5.4.1. *The difference quotient map*

$$Q : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{C}), \quad \lambda \mapsto Q(\lambda)$$

with

$$Q(\lambda)(\theta, \tau) := \frac{\lambda(\theta) - \lambda(\tau)}{e^{i\theta} - e^{i\tau}}$$

is linear-tame, and hence smooth-tame.

Proof. The strategy of the proof is to find a series of real constants $(K_n)_{n \in \mathbb{N}}$ such that $\|Q.\lambda\|_n \leq K_n \, \|\lambda\|_n$ holds for all $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$. Here $\| \cdot \|_n$ denotes the grading on $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ respectively $\mathcal{C}^\infty(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{C})$ defined in 5.1.8. For the rest of the proof, fix a natural number $n \in \mathbb{N}$ and a function $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$. Let $l, m \in \mathbb{N}_0$ such that $l + m \leq n$, and let

$$q(\theta, \tau) := \frac{\lambda(\theta) - \lambda(\tau)}{\theta - \tau} = \int_0^1 \lambda'(\xi \cdot \theta + (1 - \xi) \cdot \tau) \, d\xi.$$

Then we have

$$\begin{aligned}
|\left(\frac{\partial}{\partial\theta}\right)^l \left(\frac{\partial}{\partial\tau}\right)^m q(\theta, \tau)| &= |\int_0^1 \left(\frac{\partial}{\partial\theta}\right)^l \left(\frac{\partial}{\partial\tau}\right)^m \lambda'(\xi \cdot \theta + (1 - \xi) \cdot \tau) \, d\xi| \\
&= |\int_0^1 \xi^l (1 - \xi)^m \lambda^{(l+m+1)}(\xi \cdot \theta + (1 - \xi) \cdot \tau) d\xi| \\
&\leq \mathsf{sup}_{t \in \mathbb{S}^1} |\lambda^{(l+m+1)}(t)| \\
&\leq \sum_{j=0}^{l+m+1} \mathsf{sup}_{t \in \mathbb{S}^1} |\lambda^{(j)}(t)| \\
&\leq \|\lambda\|_{n+1}
\end{aligned}$$

for all $\theta, \tau \in \mathbb{R}$. For a moment, consider the case $|\theta - \tau| \leq \frac{3}{2}\pi$. Then

$$\eta(\theta, \tau) := \frac{\theta - \tau}{e^{i\theta} - e^{i\tau}}$$

depends smoothly on θ and τ. Therefore,

$$E_n := \sum_{l,m=0}^{n} \mathsf{sup}_{|\theta-\tau|\leq\frac{3}{2}\pi} \left| \left(\frac{\partial}{\partial\theta}\right)^l \left(\frac{\partial}{\partial\tau}\right)^m \eta(\theta, \tau) \right|,$$

exists and we have

$$\left(\frac{\partial}{\partial\theta}\right)^l \left(\frac{\partial}{\partial\tau}\right)^m \eta(\theta, \tau) \leq E_n.$$

Moreover, we have

$$\begin{aligned} [Q.\lambda](\theta, \tau) &= \frac{\lambda(\theta) - \lambda(\tau)}{e^{i\theta} - e^{i\tau}} \\ &= \frac{\lambda(\theta) - \lambda(\tau)}{\theta - \tau} \cdot \frac{\theta - \tau}{e^{i\theta} - e^{i\tau}} \\ &= q(\theta, \tau) \cdot \eta(\theta, \eta), \end{aligned}$$

and therefore

$$\begin{aligned} |\left(\frac{\partial}{\partial\theta}\right)^l \left(\frac{\partial}{\partial\tau}\right)^m [Q.\lambda](\theta, \tau)| &= |\left(\frac{\partial}{\partial\theta}\right)^l \left(\frac{\partial}{\partial\tau}\right)^m q(\theta, \tau) \cdot \eta(\theta, \eta)| \\ &\leq 4^n \cdot E_n \cdot \|\lambda\|_{n+1} \qquad (\dagger) \end{aligned}$$

for $l + m \leq n$. Since $[Q.\lambda](\theta, \tau)$ is 2π-periodic in the variables θ and τ, and $(\dagger)$ is shown for $|\theta - \tau| < \frac{3}{2}\pi$ the inequality $(\dagger)$ holds for all $\theta, \tau \in \mathbb{R}$. This implies

$$\begin{aligned} \|Q.\lambda\|_n &\overset{5.1.8}{=} \sum_{k=0}^{n} \sum_{j=0}^{k} \mathsf{sup}_{\theta,\tau\in\mathbb{S}^1} \left| \left(\frac{\partial}{\partial\theta}\right)^{k-j} \left(\frac{\partial}{\partial\tau}\right)^j [Q.\lambda](\theta, \tau) \right| \\ &\leq \sum_{k=0}^{n} \sum_{j=0}^{k} 4^n \cdot E_n \cdot \|\lambda\|_{n+1} \\ &\leq \frac{1}{2}(n+2)(n+1) \cdot 4^n \cdot E_n \cdot \|\lambda\|_{n+1}, \end{aligned}$$

and we have found $K_n = \frac{1}{2}(n+2)(n+1)\, 4^n\, E_n$. According to Definition 5.1.2, we have shown that Q is linear-tame, and hence it is by Lemma 5.2.7 smooth-tame. $\square$

Lemma 5.4.2. *The map*

$$H : \mathtt{V} \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{C}), \quad (f, \lambda) \mapsto H(f, \lambda)$$

with

$$[H(f, \lambda)](\theta, \tau) = \frac{\lambda(\theta) - \lambda(\tau)}{f(\theta) - f(\tau)} f'(\theta)$$

is smooth-tame.

Proof. Since $f \in \mathtt{V}$ is injective, the difference quotient

$$[Q.f](\theta, \tau) = \frac{f(\theta) - f(\tau)}{e^{i\theta} - e^{i\tau}}$$

is never zero. So, we can write H as a product

$$\frac{\lambda(\theta) - \lambda(\tau)}{f(\theta) - f(\tau)} f'(\theta) = [(Q.\lambda)(\theta, \tau)] \cdot [\frac{1}{(Q.f)(\theta, \tau)}] \cdot [f'(\theta)]$$

of three maps. The first map $\lambda \mapsto Q.\lambda$ is smooth-tame by Lemma 5.4.1. Let

$$\rho : \mathbb{C}^\times \to \mathbb{C}^\times, \quad z \mapsto \frac{1}{z},$$

then

$$\mathsf{L}_\rho : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times)$$

is smooth-tame by Lemma 5.3.5. This yields that the second map

$$\mathtt{V} \to \mathcal{C}^\infty(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{C}^\times), \quad f \mapsto \frac{1}{Q.f} = \mathsf{L}_\rho.Q.f$$

is also smooth-tame. The third map is just the derivative map, which is smooth-tame by Lemma 5.2.10(i). $\square$

Lemma 5.4.3. *The integration map*

$$I : \mathcal{C}^\infty(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \quad h \mapsto I(h)$$

with

$$I(h)(\tau) := \int_{\mathbb{S}^1} h(\theta, \tau) d\theta\}$$

is linear-tame, and hence smooth-tame.

Proof. We have

$$\begin{aligned}
\|I.h\|_n = & \sum_{j=0}^{n} \mathsf{sup}_{\theta\in\mathbb{S}^1} \|(I.h)^{(j)}(\theta)\| \\
= & \sum_{j=0}^{n} \mathsf{sup}_{\theta\in\mathbb{S}^1} \|\int_{\mathbb{S}^1} \partial_\theta^{(j)} h(\theta,\tau)d\tau\| \\
\leq & \sum_{j=0}^{n} \mathsf{sup}_{\theta\in\mathbb{S}^1}\, 2\pi\cdot \mathsf{sup}_{\tau\in\mathbb{S}^1}\|\partial_\theta^{(j)} h(\theta,\tau)\| \\
\leq & \sum_{j=0}^{n}\sum_{j=0}^{k} \mathsf{sup}_{\theta,\tau\in\mathbb{S}^1}\|\partial_\tau^{k-j}\partial_\theta^{j} h(\theta,\tau)\| \\
\leq & \ 2\pi\cdot\|h\|_n.
\end{aligned}$$

for all $h \in \mathcal{C}^\infty(\mathbb{S}^1\times\mathbb{S}^1,\mathbb{C})$, and according to Definition 5.1.2, I is linear-tame.

□

Proposition 5.4.4. *The map*

$$P : \mathtt{V}\times\mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}), \quad (f,\lambda)\mapsto P(f,\lambda)$$

with

$$[P(f,\lambda)](\tau) := \frac{i}{2\pi}\int_{\mathbb{S}^1} \frac{\lambda(\theta)-\lambda(\tau)}{f(\theta)-f(\tau)} f'(\theta)d\theta$$

is smooth-tame.

Proof. The map P is the composition of the smooth-tame map

$$H : \mathtt{V}\times\mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1\times\mathbb{S}^1,\mathbb{C})$$

of Lemma 5.4.2 and the smooth-tame integration map

$$I : \mathcal{C}^\infty(\mathbb{S}^1\times\mathbb{S}^1,\mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C})$$

of Lemma 5.4.3 multiplied with the real constant $i/2\pi$. Since by Lemma 5.2.8 the composition of smooth-tame is smooth-tame, the lemma is proved.

□

5.5 Derivatives

In this section, we provide derivatives of some maps on function spaces to be used in Chapters 7 and 8.

Lemma 5.5.1. *Let $X \subseteq \mathbb{R}^k$, $Z \subseteq \mathbb{R}^l$, $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ be open subsets and $\rho : A \to B$ a smooth map. Then*

(i) *the derivative of*

$$\mathsf{L}_\rho : \mathcal{C}^\infty(X, A) \to \mathcal{C}^\infty(X, B), \quad \gamma \mapsto \mathsf{L}_\rho(\gamma) := \rho \circ \gamma$$

is given by

$$d\mathsf{L}_\rho(\gamma, \Delta\gamma)(t) = d\rho(\gamma(t), \Delta\gamma(t)\,),$$

(ii) *and the derivative of*

$$\mathsf{R}_\rho : \mathcal{C}^\infty(B, Z) \to \mathcal{C}^\infty(A, Z), \quad f \mapsto \mathsf{R}_\rho(f) := f \circ \rho$$

is given by

$$d\mathsf{R}_\rho(f, \Delta f)(t) = [\Delta f \circ \rho](t).$$

Proof. (i):

$$\begin{aligned} d\mathsf{L}_\rho(\gamma, \Delta\gamma)(t) = & \lim_{\epsilon \to 0} \frac{1}{\epsilon} \{\mathsf{L}_\rho(\gamma + \epsilon\Delta\gamma)(t) - \mathsf{L}_\rho(\gamma)(t)\} \\ = & \lim_{\epsilon \to 0} \frac{1}{\epsilon} \{\rho(\gamma(t) - \epsilon\Delta\gamma(t)) - \rho(\gamma(t))\} \\ = & \rho'(\gamma(t)) \cdot \Delta\gamma(t). \end{aligned}$$

(ii):

$$\begin{aligned} d\mathsf{R}_\rho(f, \Delta f)(t) &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \{\mathsf{R}_\rho(f + \epsilon\Delta f)(t) - \mathsf{R}_\rho(f)(t)\} \\ &= (f + \Delta f)(\rho(t)) - f(\rho(t)). \end{aligned}$$

□

Let us say some words about the tangent space of $\mathsf{Diff}^+\mathbb{S}^1$. By Definition 2.1.3 the group $\widetilde{\mathsf{Diff}}^+\mathbb{S}^1$ is considered as a set of smooth functions $f : \mathbb{R} \to \mathbb{R}$ with $f(x + 2\pi) = f(x) + 2\pi$ and $f'(t) > 0$ for all $t \in \mathbb{S}^1$. The difference of such two functions is an element of $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$. In this way, we consider $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$ as the tangent space of $\widetilde{\mathsf{Diff}}^+\mathbb{S}^1$. Since $\widetilde{\mathsf{Diff}}^+\mathbb{S}^1$ is the simple connected covering of $\mathsf{Diff}^+\mathbb{S}^1$, we consider $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$ as the tangent space of $\mathsf{Diff}^+\mathbb{S}^1$ in the same way.

Lemma 5.5.2. *The derivative of the group inversion*

$$\iota : \mathsf{Diff}^+\mathbb{S}^1 \to \mathsf{Diff}^+\mathbb{S}^1, \quad \varphi \mapsto \iota(\varphi) = \varphi^{-1}$$

is given by

$$d\iota : \mathsf{Diff}^+\mathbb{S}^1 \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}), \quad (\varphi, \Delta\varphi) \mapsto d\iota(\varphi, \Delta\varphi)$$

with

$$[d\iota(\varphi, \Delta\varphi)](t) = -d[\varphi^{-1}](t, \Delta\varphi(\varphi^{-1}(t))),$$

for $t \in \mathbb{S}^1$.

Proof. See [13] Example 4.4.6 on page 92. □

Lemma 5.5.3. *Consider two smooth functions* $\lambda : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathsf{Diff}^+\mathbb{S}^1$ *and* $\beta : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$. *Then the derivative of*

$$\gamma : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathsf{Diff}^+\mathbb{S}^1, \quad \eta \to \gamma(\eta) := \lambda(\eta) \circ \beta(\eta)$$

is given by

$$d\gamma(\eta, \Delta\eta)(t) = d\lambda(\eta, \Delta\eta)(\beta(\eta)(t)\,) + d[\lambda(\eta)](\beta(\eta)(t), d\beta(\eta, \Delta\eta)(t)\,)$$

with $t \in \mathbb{S}^1$.

Proof. Since $\widetilde{\mathsf{Diff}}^+\mathbb{S}^1$ is the simply connected covering of $\mathsf{Diff}^+\mathbb{S}^1$, it is sufficient to show the statement for a function $\lambda : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \widetilde{\mathsf{Diff}}^+\mathbb{S}^1$. We have

$$\begin{aligned}
&d\gamma(\eta, \Delta\eta)(t) \\
=&\lim_{\epsilon\to 0} \frac{1}{\epsilon}\left\{\gamma(\eta + \epsilon\Delta\eta)(t) - \gamma(\eta)(t)\right\} \\
=&\lim_{\epsilon\to 0} \frac{1}{\epsilon}\{\lambda(\eta + \epsilon\Delta\eta)(\beta(\eta + \epsilon\Delta\eta)(t)) - \lambda(\eta)(\beta(\eta + \epsilon\Delta\eta)(t)) \\
&\qquad + \lambda(\eta)(\beta(\eta + \epsilon\Delta\eta)(t)) - \gamma(\eta)\} \\
=&\lim_{\epsilon\to 0} \frac{1}{\epsilon}\left\{\lambda(\eta + \epsilon\Delta\eta)(\beta(\eta + \epsilon\Delta\eta)(t)) - \lambda(\eta)(\beta(\eta + \epsilon\Delta\eta)(t))\right\} \\
&+ \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left\{\lambda(\eta)(\beta(\eta + \epsilon\Delta\eta)(t)) - \lambda(\eta)(\beta(\eta)(t))\right\} \\
=&d\lambda(\eta, \Delta\eta)(\beta(\eta)(t)) \\
&+ \lim_{\epsilon\to 0} \frac{\left\{\lambda(\eta)(\beta(\eta + \epsilon\Delta\eta)(t)) - \lambda(\eta)(\beta(\eta)(t))\right\}}{\beta(\eta + \epsilon\Delta\eta)(t) - \beta(\eta)(t)} \cdot \frac{1}{\epsilon}\left\{\beta(\eta + \epsilon\Delta\eta)(t) - \beta(\eta)(t)\right\} \\
=&d\lambda(\eta, \Delta\eta)(\beta(\eta)(t)) + d[\lambda(\eta)](\beta(\eta)(t), d\beta(\eta, \Delta\eta)(t)).
\end{aligned}$$

□

Lemma 5.5.4. *Consider two smooth functions* $\lambda : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ *and* $\beta : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathsf{Diff}^+\mathbb{S}^1$. *Then the derivative of*

$$\Xi : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathsf{Diff}^+\mathbb{S}^1$$

with

$$\Xi(\eta) := \iota(\beta(\lambda(\eta))) \circ \beta(\eta) = \beta(\lambda(\eta))^{-1} \circ \beta(\eta)$$

is

$$\begin{aligned} d\Xi(\eta, \Delta\eta)(t) = d[\beta(\lambda(\eta))^{-1}]_{\beta(\eta)(t)} \times \\ \{d\beta(\eta, \Delta\eta)(t) - d\beta(\lambda(\eta), d\lambda(\eta, \Delta\eta)\,)(\Xi(\eta)(t))\}\,. \end{aligned}$$

Proof. With the abbreviation $\alpha = \beta \circ \lambda$ we compute

$$\begin{aligned} & d\Xi(\eta, \Delta\eta)(t) \\ \overset{5.5.3}{=} \; & d[\iota \circ \alpha](\eta, \Delta\eta)(\beta(\eta)(t)) + d[\alpha(\eta)^{-1}](\beta(\eta)(t), d\beta(\eta, \Delta\eta)(t)) \\ = \; & d\iota(\alpha(\eta), d\alpha(\eta, \Delta\eta))(\beta(\eta)(t)) + d[\alpha(\eta)^{-1}]_{\beta(\eta)(t)} \,.\, d\beta(\eta, \Delta\eta)(t) \\ \overset{5.5.2}{=} \; & -d[\alpha(\eta)^{-1}](\beta(\eta)(t), d\alpha(\eta, \Delta\eta)(\alpha(\eta)^{-1}(\beta(\eta)(t))\,)) \\ & + d[\alpha(\eta)^{-1}]_{\beta(\eta)(t)} \,.\, d\beta(\eta, \Delta\eta)(t) \\ = \; & d[\alpha(\eta)^{-1}]_{\beta(\eta)(t)} . \{-d\alpha(\eta, \Delta\eta)(\Xi(\eta)(t)) + d\beta(\eta, \Delta\eta)(t)\} \\ = \; & d[\alpha(\eta)^{-1}]_{\beta(\eta)(t)} . \{d\beta(\eta, \Delta\eta)(t) - d\alpha(\eta, \Delta\eta)(\Xi(\eta)(t))\} \\ = \; & d[\beta(\lambda(\eta))^{-1}]_{\beta(\eta)(t)} . \{d\beta(\eta, \Delta\eta)(t) - d[\beta \circ \lambda](\eta, \Delta\eta)(\Xi(\eta)(t))\} \\ = \; & d[\beta(\lambda(\eta))^{-1}]_{\beta(\eta)(t)} . \{d\beta(\eta, \Delta\eta)(t) - d\beta(\lambda(\eta), d\lambda(\eta, \Delta\eta)\,)(\Xi(\eta)(t))\}\,. \end{aligned}$$

□

Lemma 5.5.5. *Let*

$$\lambda : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$$

and

$$\beta : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathsf{Diff}^+\mathbb{S}^1$$

be two smooth functions. Then the derivative of

$$\alpha : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathsf{Diff}^+\mathbb{S}^1, \quad \eta \to \alpha(\eta) = \lambda(\eta) \circ \underbrace{\iota(\beta(\eta)\,)}_{\beta(\eta)^{-1}}$$

reads

$$d\alpha(\eta, \Delta\eta)(t) = d\lambda(\eta, \Delta\eta)(\beta(\eta)^{-1}(t)) - d[\alpha(\eta)]_t . d\beta(\eta, \Delta\eta)(\beta(\eta)^{-1}(t)).$$

Proof.

$$
\begin{aligned}
& d\alpha(\eta,\Delta\eta)(t) \\
\overset{5.5.3}{=}\ & d\lambda(\eta,\Delta\eta)((\iota\circ\beta)(\eta)(t)) + d[\lambda(\eta)]((\iota\circ\beta)(\eta)(t), d[\iota\circ\beta](\eta,\Delta\eta)(t)) \\
=\ & d\lambda(\eta,\Delta\eta)(\beta(\eta)^{-1}(t)) + d[\lambda(\eta)](\beta(\eta)^{-1}(t), d[\iota\circ\beta](\eta,\Delta\eta)(t)) \\
=\ & d\lambda(\eta,\Delta\eta)(\beta(\eta)^{-1}(t)) + d[\lambda(\eta)](\beta(\eta)^{-1}(t),\ d\iota(\beta(\eta),\ d\beta(\eta,\Delta\eta))(t)\ ) \\
\overset{5.5.2}{=}\ & d\lambda(\eta,\Delta\eta)(\beta(\eta)^{-1}(t)) \\
& + d[\lambda(\eta)](\beta(\eta)^{-1}(t),\ -d[\beta(\eta)^{-1}](t, d\beta(\eta,\Delta\eta)(\beta(\eta)^{-1}(t)))\ ) \\
=\ & d\lambda(\eta,\Delta\eta)(\beta(\eta)^{-1}(t)) - d[\lambda(\eta)]_{\beta(\eta)^{-1}(t)} \,.\, d[\beta(\eta)^{-1}]_t \,.\, d\beta(\eta,\Delta\eta)(\beta(\eta)^{-1}(t)) \\
=\ & d\lambda(\eta,\Delta\eta)(\beta(\eta)^{-1}(t)) - d[\alpha(\eta)]_t . d\beta(\eta,\Delta\eta)(\beta(\eta)^{-1}(t)).
\end{aligned}
$$

$\square$

Chapter 6

Tubular Neighborhood

Consider a diffeomorphic embedding $f : \mathbb{S}^1 \to \mathbb{C}^\times$ with $f \in \mathtt{V}$ and consider the image $f(\mathbb{S}^1)$ as a submanifold of $\mathbb{C}^\times$. If $\mathtt{r}_f > 0$ is sufficiently small, the function

$$\mathtt{N}_f : (-\mathtt{r}_f, \mathtt{r}_f) \times \mathbb{S}^1 \to \mathbb{C}^\times, \quad (\xi, t) \mapsto f(t) - i\frac{f'(t)}{|f'(t)|} \cdot \xi$$

defines a tubular neighborhood around $f(\mathbb{S}^1)$. In particular, $\mathtt{N}_f$ is a diffeomorphic embedding and in the complex plane the coordinates ξ and t are orthogonal to each other, where ξ is the distance between a point and the submanifold $f(\mathbb{S}^1)$. This is well-known [5]. But, for our purposes we need the tube radius explicitly (Lemma 6.1.7). This is necessary to show that $\mathtt{V}$ is an open subset of $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$. In Lemma 8.5.4 the explicit formula of the tube radius will be used, for a continuity argument. Moreover, in Chapter 7, we need the tube function $\mathtt{N}_f$ in order to construct atlases of $\mathtt{V}$ and $\mathtt{V}^E$.

At the end of this chapter in section 3, we show that the subsets

$$\mathtt{V}^+, \mathtt{V}^E, \mathtt{V} \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$$

are path connected.

6.1 The Tube

In this section, we construct a tubular neighborhood, for the image $f(\mathbb{S}^1)$ and determine the width $\mathtt{r}_f$ of the tube, called the tube radius, explicitly. As a first application, we show (6.1.8) that $\mathtt{V} \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is an open subset.

Definition 6.1.1 (Tube radius). For a 2π-periodic function $f : \mathbb{S}^1 \to \mathbb{C}^\times$ with $f \in \mathtt{V}$ we define the *tube radius* to be

$$\mathtt{r}_f := \frac{1}{2}\mathsf{min}\{u, v, w\} \qquad \text{with}$$

(i) $w := \mathsf{inf}_{t\in\mathbb{R}}|f(t)|$,
(ii) $\alpha := \mathsf{inf}_{t\in\mathbb{R}}|f'(t)|$,
(iii) $\beta := \mathsf{sup}_{t\in\mathbb{R}}|f''(t)|$,[1]
(iv) $u := \frac{1}{4}\frac{\alpha^2}{\beta}$ (curvature radius),
(v) $|t_1 - t_2|_{\mathbb{S}^1} := \mathsf{min}\{\,|t_1 - t_2 - \tau| \in \mathbb{R}^+ \,:\, \tau \in 2\pi\mathbb{Z}\,\}$, and
(vi) $v := \mathsf{min}\{\,\frac{1}{2}|f(t_2) - f(t_1)| \in \mathbb{R}^+ \,:\, |t_1 - t_2|_{\mathbb{S}^1} \geq \frac{1}{2}\frac{\alpha}{\beta}\,\}$.

Remark: The inequality $\mathtt{r}_f < \frac{1}{2}w$ guarantees that the tube does not meet $0 \in \mathbb{C}$, and $\mathtt{r}_f < \frac{1}{2}u$ secures the injectivity for small distances (see 2. Case of 6.1.5) and $\mathtt{r}_f < \frac{1}{2}v$ the injectivity for large distances (see 3. Case of 6.1.5).

Lemma 6.1.2. $\mathtt{r}_f > 0$.

Proof. The values of w and α are greater than zero because $\mathbb{S}^1$ is compact and $f'(t), f(t) \neq 0$ for every $t \in \mathbb{S}^1$. Moreover, since f is 2π-periodic but not constant we have $\beta > 0$. Hence, $u > 0$. Since $\alpha > 0$, and the function $f \in \mathtt{V}$ is injective we have $v > 0$. We conclude that $\mathtt{r}_f := \frac{1}{2}\mathsf{min}\{u, v, w\} > 0$. □

Definition 6.1.3 (Tube function)**.** For $f \in \mathtt{V}$ we define

$$\mathtt{N}_f(\xi, t) := f(t) - i\frac{f'(t)}{|f'(t)|} \cdot \xi$$

with $\xi \in \mathbb{R}$ and $t \in \mathbb{S}^1$. Furthermore, we write

$$\mathtt{A}_f := \mathtt{N}_f((-\mathtt{r}_f, \mathtt{r}_f) \times \mathbb{S}^1) \subseteq \mathbb{C}$$

for the image.

Remark: Note that we consider $\mathtt{N}_f$ as a function with domain $(-\mathtt{r}_f, \mathtt{r}_f) \times \mathbb{S}^1$ as well as a 2π-periodic function with domain $(-\mathtt{r}_f, \mathtt{r}_f) \times \mathbb{R}$.

Lemma 6.1.4. $0 \notin \mathtt{A}_f$.

Proof.

$$\begin{aligned} |\mathtt{N}_f(\xi, t)| &\geq |f(t)| - |\xi| \geq |f(t)| - \mathtt{r}_f \geq |f(t)| - \frac{1}{2}w \\ &= |f(t)| - \frac{1}{2}\mathsf{inf}_{\theta\in\mathbb{R}}|f(\theta)| \geq \frac{1}{2}|f(t)| > 0. \end{aligned}$$

□

[1] Note that $\beta > 0$, which is needed for the definition of u.

Lemma 6.1.5. *For $f \in \mathtt{V}$ the function*

$$\mathtt{N}_f : (-\mathtt{r}_f, \mathtt{r}_f) \times \mathbb{S}^1 \to \mathbb{C}, \quad (\xi, t) \mapsto \mathtt{N}_f(\xi, t)$$

defined in 6.1.3 is injective.

Proof. Let $t_1, t_2 \in \mathbb{R}$ and $\xi_1, \xi_2 \in (-\mathtt{r}_f, \mathtt{r}_f)$. We assume without loss of generality that $|t_1 - t_2|_{\mathbb{S}^1} = |t_1 - t_2|$, becasue $\mathtt{N}_f$ is 2π-periodic in the second variable t. We have to show $\mathtt{N}_f(\xi_2, t_2) - \mathtt{N}_f(\xi_1, t_1) \neq 0$ and distinguish three cases:

1. Case - $|t_2 - t_1| = |t_2 - t_1|_{\mathbb{S}^1} = 0$ and $\xi_2 \neq \xi_1$:
Let $t = t_1 = t_2$, then

$$\mathtt{N}_f(\xi_2, t_2) - \mathtt{N}_f(\xi_1, t_1) = \mathtt{N}_f(\xi_2, t) - \mathtt{N}_f(\xi_1, t) = -i\frac{f'(t)}{|f'(t)|}(\xi_2 - \xi_1) \neq 0.$$

2. Case - $0 < |t_2 - t_1| = |t_2 - t_1|_{\mathbb{S}^1} < \frac{1}{2}\frac{\alpha}{\beta}$:

$$\begin{aligned}
&\mathtt{N}_f(\xi_2, t_2) - \mathtt{N}_f(\xi_1, t_1) \\
= \ & [f(t_2) - f(t_1)] - i\left(\frac{f'(t_2)}{|f'(t_2)|} \cdot \xi_2 - \frac{f'(t_1)}{|f'(t_1)|} \cdot \xi_1\right) \\
= \ & (t_2 - t_1) \cdot f'(t_1) + \int_{t_1}^{t_2} (f'(\tau) - f'(t_1))\, d\tau \\
& \qquad - i\left(\frac{\xi_2}{|f'(t_2)|} - \frac{\xi_1}{|f'(t_1)|}\right) f'(t_1) - i\frac{\xi_2}{|f'(t_2)|}[f'(t_2) - f'(t_1)] \\
= \ & (t_2 - t_1) \cdot f'(t_1) + \int_{t_1}^{t_2} \int_{t_1}^{\tau} f''(\theta)\, d\theta\, d\tau \\
& \qquad - i\left(\frac{\xi_2}{|f'(t_2)|} - \frac{\xi_1}{|f'(t_1)|}\right) f'(t_1) - i\frac{\xi_2}{|f'(t_2)|} \int_{t_1}^{t_2} f''(\theta)\, d\theta \\
= \ & \left((t_2 - t_1) - i[\frac{\xi_2}{|f'(t_2)|} - \frac{\xi_1}{|f'(t_1)|}]\right) f'(t_1) \\
& \qquad + \int_{t_1}^{t_2} \int_{t_1}^{\tau} f''(\theta) d\theta d\tau - i\frac{\xi_2}{|f'(t_2)|} \int_{t_1}^{t_2} f''(\theta)\, d\theta.
\end{aligned}$$

Using the triangle inequality, we estimate

$$
\begin{aligned}
&|\mathtt{N}_f(\xi_2, t_2) - \mathtt{N}_f(\xi_1, t_1)| \\
\geq &\left| \left((t_2 - t_1) - i[\frac{\xi_2}{|f'(t_2)|} - \frac{\xi_1}{|f'(t_1)|}] \right) \right| \cdot |f'(t_1)| \\
&\qquad - \left| \int_{t_1}^{t_2} \int_{t_1}^{\tau} f''(\theta) d\theta d\tau \right| - \left| \frac{\xi_2}{|f'(t_2)|} \int_{t_1}^{t_2} f''(\theta) d\theta \right| \\
\geq &|t_2 - t_1| \cdot |f'(t_1)| - \frac{1}{2}|t_2 - t_1|^2 \cdot \mathsf{sup}_{\theta \in \mathbb{R}} |f''(\theta)| \\
&\qquad - \frac{|\xi_2|}{|f'(t_2)|} |t_2 - t_1| \cdot \mathsf{sup}_{\theta \in \mathbb{R}} |f''(\theta)| \\
&\qquad [\text{ using } \beta := \mathsf{sup}_{\theta \in \mathbb{R}} |f''(\theta) \ ] \\
\geq &|t_2 - t_1| \cdot |f'(t_1)| - \frac{1}{2}|t_2 - t_1|^2 \cdot \beta - \frac{|\xi_2|}{|f'(t_2)|} |t_2 - t_1| \cdot \beta| \\
&\qquad [\text{ using } |t_2 - t_1| \leq \frac{1}{2}\frac{\alpha}{\beta} \ ] \\
\geq &|t_2 - t_1| \cdot |f'(t_1)| - \frac{1}{2}|t_2 - t_1| \frac{1}{2}\frac{\alpha}{\beta} \cdot \beta - \frac{|\xi_2|}{|f'(t_2)|} |t_2 - t_1| \cdot \beta \\
&\qquad [\text{ using } |\xi_2| \leq \mathtt{r}_f \leq \frac{1}{4}\frac{\alpha^2}{\beta}, \ |f'(t_2) \geq \alpha| \ ] \\
\geq &|t_2 - t_1| \cdot |f'(t_1)| - \frac{1}{2}|t_2 - t_1| \frac{1}{2}\frac{\alpha}{\beta} \cdot \beta - \frac{1}{4}\frac{\alpha}{\beta} |t_2 - t_1| \cdot \beta \\
&\qquad [\text{ using } |f'(t_1) > \alpha| \ ] \\
\geq &|t_2 - t_1| \cdot \alpha \cdot (1 - \frac{1}{4} - \frac{1}{4}) > 0.
\end{aligned}
$$

3. Case - $|t_2 - t_1|_{\mathbb{S}^1} \geq \frac{1}{2}\frac{\alpha}{\beta}$:
We have $\mathtt{r}_f \leq \frac{1}{2}v \leq \frac{1}{4}|f(t_2) - f(t_1)|$, and hence

$$
\begin{aligned}
|\mathtt{N}_f(\xi_2, t_2) - \mathtt{N}_f(\xi_1, t_1)| &\geq \ |f(t_2) - f(t_1)| - |\xi_2| - |\xi_1| \\
&\geq \ 2v - \mathtt{r}_f - \mathtt{r}_f \\
&\geq \ 4\mathtt{r}_f - \mathtt{r}_f - \mathtt{r}_f \\
&= \ 2\mathtt{r}_f > 0.
\end{aligned}
$$

□

The next lemma shows that the coordinate ξ corresponds to the distance between a point $p \in \mathtt{A}_f$ and the curve $f(\mathbb{S}^1) \subseteq \mathbb{C}^\times$.

Lemma 6.1.6. *Given a function $f \in \mathtt{V}$, we have*

$$|f(t_2) - \mathtt{N}_f(\xi, t_1)| \geq |\xi|$$

for all $\xi \in (-\mathtt{r}_f, \mathtt{r}_f)$ and $t_1, t_2 \in \mathbb{R}$.

Proof. This proof requires the explicit determination of $\mathtt{r}_f$ given in Definition 6.1.1. Let $\xi \in (-\mathtt{r}_f, \mathtt{r}_f)$ and $t_2, t_1 \in \mathbb{R}$. Assume

$$|t_2 - t_1|_{\mathbb{S}^1} = |t_2 - t_1|$$

without loss of generality, because f and $\mathtt{N}_f(\xi, \cdot)$ are 2π-periodic functions. We distinguish two cases:
1. Case - $|t_2 - t_1| = |t_2 - t_1|_{\mathbb{S}^1} < \frac{1}{2}\frac{\alpha}{\beta}$:

$$\begin{aligned}
& |f(t_2) - \mathtt{N}_f(\xi, t_1)| \\
= \; & |f(t_2) - f(t_1) + i\frac{f'(t_1)}{|f'(t_1)|} \cdot \xi| \\
= \; & |(t_2 - t_1) \cdot f'(t_1) + \int_{t_1}^{t_2} (f'(\tau) - f'(t_1)\,)\,d\tau + i\frac{f'(t_1)}{|f'(t_1)|} \cdot \xi| \\
= \; & |(t_2 - t_1) \cdot f'(t_1) + i\frac{f'(t_1)}{|f'(t_1)|} \cdot \xi + \int_{t_1}^{t_2} \int_{t_1}^{\tau} f''(\lambda) d\lambda d\tau| \\
\geq \; & |(t_2 - t_1) \cdot f'(t_1) + i\frac{f'(t_1)}{|f'(t_1)|} \cdot \xi| - |\int_{t_1}^{t_2} \int_{t_1}^{\tau} f''(\lambda) d\lambda d\tau| \\
\geq \; & |\frac{f'(t_1)}{|f'(t_1)|} \cdot ((t_2 - t_1) \cdot |f'(t_1)| + i \cdot \xi)\,| - \int_{t_1}^{t_2} \int_{t_1}^{\tau} |f''(\lambda)| d\lambda d\tau \\
& \qquad [\text{ using } \beta \geq |f''(\lambda)| \,] \\
\geq \; & |(t_2 - t_1) \cdot |f'(t_1)| + i \cdot \xi| - \frac{1}{2}|t_2 - t_1|^2 \cdot \beta \\
& \qquad [\text{ using } \alpha \leq |f'(t_1)| \,] \\
\geq \; & |(t_2 - t_1) \cdot \alpha + i \cdot \xi| - \frac{1}{2}|t_2 - t_1|^2 \cdot \beta \\
= \; & \sqrt{(t_2 - t_1)^2 \cdot \alpha^2 + \xi^2} - \frac{1}{2}|t_2 - t_1|^2 \cdot \beta \\
\geq \; & \sqrt{\frac{1}{4}|t_2 - t_1|^2 \alpha^2 + |t_2 - t_1|^2 \frac{1}{8}\alpha^2 + \xi^2} - \frac{1}{2}|t_2 - t_1|^2 \cdot \beta
\end{aligned}$$

[using $|t_2 - t_1|^2 \leq \frac{\alpha^2}{\beta^2}$]

$$\geq \sqrt{\frac{1}{4}|t_2 - t_1|^2 \cdot |t_2 - t_1|^2 \beta^2 + |t_2 - t_1|^2 \frac{1}{8}\alpha^2 + \xi^2} - \frac{1}{2}|t_2 - t_1|^2 \cdot \beta$$

[using $|\xi| \leq \mathtt{r}_f \leq \frac{1}{8}\frac{\alpha^2}{\beta}$]

$$\begin{aligned}
&\geq \sqrt{\frac{1}{4}|t_2 - t_1|^2 \cdot |t_2 - t_1|^2 \beta^2 + |t_2 - t_1|^2 \cdot \beta|\xi| + \xi^2} - \frac{1}{2}|t_2 - t_1|^2 \cdot \beta \\
&\geq \sqrt{\left[\frac{1}{2}|t_2 - t_1|^2 \beta + \xi\right]^2} - \frac{1}{2}|t_2 - t_1|^2 \cdot \beta \\
&= |\xi|
\end{aligned}$$

2. Case - $|t_2 - t_1| = |t_2 - t_1|_{\mathbb{S}^1} \geq \frac{1}{2}\frac{\alpha}{\beta}$:

$$\begin{aligned}
|f(t_2) - \mathtt{N}_f(\xi, t_1)| &\geq |f(t_2) - f(t_1)| - |\xi| \geq 2v - \mathtt{r}_f \\
&\geq 4\mathtt{r}_f - \mathtt{r}_f \geq \mathtt{r}_f \geq |\xi|.
\end{aligned}$$

□

Remark: Consider ξ and t as coordinates in the complex plane. From the lemma above, we observe that ξ is the distance between a point and the submanifold $f(\mathbb{S}^1)$.

For the following lemmas, we need the explicit estimation of the tube radius $\mathtt{r}_f$.

Lemma 6.1.7. *Let $f \in \mathtt{V}$, $\Delta f \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ with $|\Delta f(t)| \leq \frac{1}{2}\mathtt{r}_f$ and $|\Delta f'(t)| \leq \frac{1}{2}\alpha$, where*

$$\alpha = \mathsf{inf}_{t \in \mathbb{R}}|f'(t)| > 0$$

as defined in 6.1.1. Then $f + \Delta f : \mathbb{S}^1 \to \mathbb{C}$ is injective. [2]

Proof. Let $\xi \in (-\mathtt{r}_f, \mathtt{r}_f)$ and $t_1, t_2 \in \mathbb{R}$ and assume

$$|t_1 - t_2|_{\mathbb{S}^1} = |t_1 - t_2|$$

without loss of generality, because f and Δf are 2π-periodic functions. Throughout the calculation we use Definition 6.1.1 and distinguish two

[2] Δ is **not** the Laplace operator, but indicates that Δf lies in the tangent space of V.

cases.

1. Case - $|t_2 - t_1| = |t_2 - t_1|_{\mathbb{S}^1} \leq \frac{1}{2}\frac{\alpha}{\beta}$:

$$
\begin{aligned}
& |(f+\Delta f)(t_2) - (f+\Delta f)(t_1)| \\
\geq\ & |f(t_2) - f(t_1)| - |\Delta f(t_2) - \Delta f(t_1)| \\
\geq\ & |(t_2 - t_1) \cdot f'(t_1) + \int_{t_1}^{t_2} \int_{t_1}^{\tau} f''(\theta) d\theta \, d\tau| - |\int_{t_1}^{t_2} \Delta f'(\theta) d\theta| \\
\geq\ & |t_2 - t_1| \cdot |f'(t_1)| - \frac{1}{2}|t_2 - t_1|^2 \cdot \mathsf{sup}|f''| - |t_2 - t_1| \cdot \mathsf{sup}|\Delta f'| \\
\geq\ & |t_2 - t_1| \cdot \mathsf{inf}|f'| - \frac{1}{2} \cdot \frac{1}{2}\frac{\alpha}{\beta} \cdot |t_2 - t_1| \cdot \beta - |t_2 - t_1| \cdot \frac{\alpha}{2} \\
\geq\ & |t_2 - t_1| \cdot (\alpha - \frac{\alpha}{4} - \frac{\alpha}{2}) \\
\geq\ & |t_2 - t_1| \cdot \frac{\alpha}{4}
\end{aligned}
$$

2. Case - $|t_2 - t_1| = |t_2 - t_1|_{\mathbb{S}^1} \geq \frac{1}{2}\frac{\alpha}{\beta}$:

$$
\begin{aligned}
|(f+\Delta f)(t_2) - (f+\Delta f)(t_1)| \geq\ & |f(t_2) - f(t_1)| - |\Delta f(t_1)| - |\Delta f(t_2)| \\
\geq\ & |f(t_2) - f(t_1)| - \frac{1}{2}\mathtt{r}_f - \frac{1}{2}\mathtt{r}_f \\
\geq\ & 2v - \frac{1}{2}\mathtt{r}_f - \frac{1}{2}\mathtt{r}_f \\
\geq\ & 4\mathtt{r}_f - \frac{1}{2}\mathtt{r}_f - \frac{1}{2}\mathtt{r}_f = 3\mathtt{r}_f \\
\neq\ & 0.
\end{aligned}
$$

$\square$

Proposition 6.1.8. *The subset* $\mathtt{V} \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ *as defined in Definition 2.4.3 is open with respect to the Fréchet topology generated by the series of seminorms* $(\|\cdot\|_n)_{n\in\mathbb{N}}$ *with*

$$\|f\|_n := \mathsf{sup}_{t\in\mathbb{S}^1}|f^{(n)}(t)|_n$$

for $f \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$.

Proof. By Lemma 5.2.9, the subset

$$V_1 := \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times) := \{ f \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \, : \, \forall_{t \in \mathbb{S}^1} \; f(t) \neq 0 \}$$

is an open subset of $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$, because $\mathbb{C}^\times$ is open in $\mathbb{C}$. The derivative map

$$\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad f \to f'$$

is continuous by Lemma 5.2.10(i), and therefore

$$V_2 := \{ f \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \, : \, \forall_{t \in \mathbb{S}^1} \; f'(t) \neq 0 \}$$

is an open subset of $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$, because it is the preimage of V_1. The winding number function

$$\mathtt{w} : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times) \to \mathbb{Z}, \quad f \mapsto \frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{f'(t)}{f(t)} dt$$

is continuous by Lemma 5.3.7, which implies that the preimage

$$V_3 := \mathtt{w}^{-1}(1)$$

is an open subset of $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times)$. We conclude

$$W := V_1 \cap V_2 \cap V_3 \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$$

is open. For $f \in \mathtt{V}$ Lemma 6.1.7 states that all functions of the open neighborhood

$$\mathsf{B}_{\frac{1}{2}\mathtt{r}_f}(f) = \{ g \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \, : \, \mathsf{sup}_{t \in \mathbb{S}^1} |g - f| < \frac{1}{2}\mathtt{r}_f \}$$

are injective. Therefore, $U_f := W \cap \mathsf{B}_{\frac{1}{2}\mathtt{r}_f}(f)$ is an open neighborhood of f, and $U_f \subseteq \mathtt{V}$. We conclude V is an open subset of $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$. $\square$

Definition 6.1.9. We define

(i) $\pi_1 : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}, \quad (\xi, t) \mapsto \xi$, and

(ii) $\pi_2 : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}, \quad (\xi, t) \mapsto t$.

Lemma 6.1.10. *If $f \in R$, then*

(i) $\pi_1(\mathtt{N}_f^{-1}(f(t))) = 0$ *for* $t \in \mathbb{S}^1$, *and*

(ii) $\pi_2(\mathtt{N}_f^{-1}(f(t))) = t$ *for* $t \in \mathbb{S}^1$.

Proof. This is a consequence of $\mathtt{N}_f(\xi, t) = f(t) - i\frac{f'(t)}{|f'(t)|} \cdot \xi$ Definition 6.1.3. $\square$

Lemma 6.1.11. *If $f \in \mathtt{V}$ and $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$, then*

(i) $\mathtt{N}_f(\xi, \gamma(t)) = \mathtt{N}_{f\circ\gamma}(\xi, t)$ *for* $t \in \mathbb{S}^1$ *and* $\xi \in (-\mathtt{r}_f, \mathtt{r}_f)$,

(ii) $\pi_1 \circ \mathtt{N}_f^{-1} = \pi_1 \circ \mathtt{N}_{f\circ\gamma}^{-1}$, *and*

(iii) $\pi_2 \circ \mathtt{N}_f^{-1} = \gamma \circ \pi_2 \circ \mathtt{N}_{f\circ\gamma}^{-1}$.

Proof. (i): Definition 6.1.3 yields

$$\mathtt{N}_f(\xi, \gamma(t)) := f(\gamma(t)) - i\frac{f'(\gamma(t))}{|f'(\gamma(t))|} \cdot \xi = (f \circ \gamma)(t) - i\frac{(f\circ\gamma)'(t)}{|(f\circ\gamma)'(t)|} \cdot \xi.$$

(ii)+(iii): We have

$$\begin{aligned} \mathtt{N}_f(\pi_1(\mathtt{N}_f^{-1}(z)), \pi_2(\mathtt{N}_f^{-1}(z))) &= z \\ &= \mathtt{N}_{f\circ\gamma}(\pi_1(\mathtt{N}_{f\circ\gamma}^{-1}(z)), \pi_2(\mathtt{N}_{f\circ\gamma}^{-1}(z))) \\ &\overset{(i)}{=} \mathtt{N}_f(\pi_1(\mathtt{N}_{f\circ\gamma}^{-1}(z)), (\gamma \circ \pi_2 \circ \mathtt{N}_{f\circ\gamma}^{-1})(z)) \end{aligned}$$

for $z \in \mathtt{A}_{f\circ\gamma} = \mathtt{A}_f$. Since the function $\mathtt{N}_f$ is injective by Lemma 6.1.5, we compare the arguments of

$$\mathtt{N}_f(\pi_1(\mathtt{N}_f^{-1}(z)), \pi_2(\mathtt{N}_f^{-1}(z)))$$

with the arguments of

$$\mathtt{N}_f(\pi_1(\mathtt{N}_{f\circ\gamma}^{-1}(z)), (\gamma \circ \pi_2 \circ \mathtt{N}_{f\circ\gamma}^{-1})(z)),$$

and get the result. □

6.2 Derivative of $\mathtt{N}_f$

In this section, we compute the derivative of

$$\mathtt{N}_f(\xi, t) := f(t) - i\frac{f'(t)}{|f'(t)|} \cdot \xi$$

explicitly and show that $\mathtt{N}_f : (-\mathtt{r}_f, \mathtt{r}_f) \times \mathbb{S}^1 \to \mathbb{C}$ is a diffeomorphic embedding.

Lemma 6.2.1 (Partial derivatives of $\mathtt{N}_f$)**.** *The partial derivatives of the function* $\mathtt{N}_f : (-\mathtt{r}_f, \mathtt{r}_f) \times \mathbb{S}^1 \to \mathbb{C}$ *are*

(i) $\partial_1 \mathtt{N}_f(\xi, t) := \dfrac{\partial}{\partial\xi}\mathtt{N}_f(\xi, t) = -i\dfrac{f'(t)}{|f'(t)|}$, *and*

(ii) $\partial_2 \mathtt{N}_f(\xi, t) := \dfrac{\partial}{\partial t}\mathtt{N}_f(\xi, t) = f'(t) + \dfrac{f'(t)}{|f'(t)|} \cdot \mathsf{Im}\dfrac{f''(t)}{f'(t)} \cdot \xi$ *for all* $f \in \mathtt{V}$.

Proof. (i): Obvious.
(ii): We compute

$$\begin{aligned}
\frac{\partial}{\partial t}\mathtt{N}_f(\xi,t) &= \frac{\partial}{\partial t}f(t) - i\xi\cdot\frac{\partial}{\partial t}\frac{f'(t)}{|f'(t)|}\\
&= f'(t) - i\xi\cdot\frac{1}{2}\frac{|f'(t)|}{f'(t)}\cdot\frac{\partial}{\partial t}\left\{\frac{f'(t)^2}{|f'(t)|^2}\right\}\\
&= f'(t) - i\xi\cdot\frac{1}{2}\frac{|f'(t)|}{f'(t)}\cdot\frac{\partial}{\partial t}\left\{\frac{f'(t)}{\overline{f'(t)}}\right\}\\
&= f'(t) - i\xi\cdot\frac{1}{2}\frac{|f'(t)|}{f'(t)}\cdot\frac{\overline{f'(t)}\cdot f''(t)-\overline{f''(t)}\cdot f'(t)}{[\overline{f'(t)}]^2}\\
&= f'(t) - i\xi\cdot\frac{f'(t)}{|f'(t)|}\frac{1}{2}\left[\frac{f''(t)}{f'(t)}-\frac{\overline{f''(t)}}{\overline{f'(t)}}\right]\\
&= f'(t) - i\xi\cdot i\frac{f'(t)}{|f'(t)|}\cdot\mathsf{Im}\frac{f''(t)}{f'(t)}\\
&= f'(t) + \frac{f'(t)}{|f'(t)|}\cdot\mathsf{Im}\frac{f''(t)}{f'(t)}\cdot\xi.
\end{aligned}$$

□

Remark: From the lemma above, we observe that in the complex plane ξ and t considered as coordinates are orthogonal to each other.

Lemma 6.2.2 (Jacobian determinant of $\mathtt{N}_f$)**.** *Given $f\in\mathtt{V}$ and let $\mathsf{det}\,d\mathtt{N}_f(\xi,t)$ denote the Jacobian determinant of $\mathtt{N}_f$. Then*

(i) $\mathsf{det}\,d\mathtt{N}_f(\xi,t) = |f'(t)| + \xi\cdot\mathsf{Im}\frac{f''(t)}{f'(t)}$,

(ii) $\mathsf{det}\,d\mathtt{N}_f(\xi,t) > 0$, *and*

(iii) $\partial_2\mathtt{N}_f(\xi,t) = i\partial_1\mathtt{N}_f(\xi,t)\cdot\mathsf{det}\,d\mathtt{N}_f(\xi,t)$

for all $\xi\in(-\mathtt{r}_f,\mathtt{r}_f)$ and all $t\in\mathbb{S}^1$.

Proof. (i):

$$\begin{aligned}
\mathsf{det}\,d\mathtt{N}_f(\xi,t) &= \mathsf{det}\begin{vmatrix}\mathsf{Re}\,\partial_1\mathtt{N}_f(\xi,t) & \mathsf{Re}\,\partial_2\mathtt{N}_f(\xi,t)\\ \mathsf{Im}\,\partial_1\mathtt{N}_f(\xi,t) & \mathsf{Im}\,\partial_2\mathtt{N}_f(\xi,t)\end{vmatrix}\\
&= \mathsf{Re}\,\partial_1\mathtt{N}_f(\xi,t)\cdot\mathsf{Im}\,\partial_2\mathtt{N}_f(\xi,t) - \mathsf{Re}\,\partial_2\mathtt{N}_f(\xi,t)\cdot\mathsf{Im}\,\partial_1\mathtt{N}_f(\xi,t)\\
&= \mathsf{Im}(\partial_1\overline{\mathtt{N}_f(\xi,t)}\cdot\partial_2\mathtt{N}_f(\xi,t))\\
&\overset{6.2.1}{=} \mathsf{Im}(i\frac{|f'(t)|}{f'(t)}(f'(t)+\frac{f'(t)}{|f'(t)|}\cdot\mathsf{Im}\frac{f''(t)}{f'(t)}\cdot\xi))\\
&= |f'(t)| + \xi\cdot\mathsf{Im}\frac{f''(t)}{f'(t)}.
\end{aligned}$$

(ii):

$$
\begin{aligned}
\det \mathtt{N}_f(\xi,t) &\geq |f'(t)| - |\xi|\cdot|\mathrm{Im}\frac{f''(t)}{f'(t)}| \\
&\geq |f'(t)| - \mathtt{r}_f \cdot |\frac{f''(t)}{f'(t)}| \\
&\overset{6.1.1}{\geq} |f'(t)| - \frac{1}{8}\frac{\alpha^2}{\beta}\cdot|\frac{f''(t)}{f'(t)}| \\
&\overset{6.1.1(ii)+(iii)}{\geq} |f'(t)| - \frac{1}{8}\frac{(\inf_{\theta\in\mathbb{S}^1}|f'(\theta)|)^2}{\sup_{\theta\in\mathbb{S}^1}|f''(\theta)|}\cdot|\frac{f''(t)}{f'(t)}| \\
&\geq |f'(t)| - \frac{1}{8}\frac{|f'(t)|^2}{|f''(t)|}\cdot|\frac{f''(t)}{f'(t)}| \\
&= \frac{7}{8}|f'(t)| > 0.
\end{aligned}
$$

(iii):

$$
\begin{aligned}
\partial_2\mathtt{N}_f(\xi,t) &= f'(t) + \frac{f'(t)}{|f'(t)|}\cdot\mathrm{Im}\frac{f''(t)}{f'(t)}\cdot\xi \\
&= \frac{f'(t)}{|f'(t)|}(|f'(t)| + \mathrm{Im}\frac{f''(t)}{f'(t)}\cdot\xi) \\
&= \frac{f'(t)}{|f'(t)|}\cdot\det d\mathtt{N}_f(\xi,t) \\
&= i\partial_1\mathtt{N}_f(\xi,t)\cdot\det d\mathtt{N}_f(\xi,t).
\end{aligned}
$$

□

Lemma 6.2.3. *If $f \in \mathtt{V}$, then*

$$\mathtt{N}_f : (-\mathtt{r}_f, \mathtt{r}_f)\times\mathbb{S}^1 \to \mathbb{C}\backslash\{0\}, \quad (\xi,t)\mapsto \mathtt{N}_f(\xi,t)$$

is a diffeomorphic embedding. Especially its image

$$\mathtt{A}_f = \mathtt{N}_f((-\mathtt{r}_f,\mathtt{r}_f)\times\mathbb{S}^1) \subseteq \mathbb{C}^\times$$

is open.

Proof. By Lemma 6.2.2(ii), the Jacobian determinant is never zero, so we can apply the Inverse Function Theorem for finite dimensions. Since $\mathtt{N}_f$ is injective by Lemma 6.1.5, the lemma is proved. □

Lemma 6.2.4. *If $f \in \mathtt{V}$, then*
(i)

$$\begin{aligned} & d[\pi_1 \circ \mathtt{N}_f^{-1}](z, \Delta z) - \zeta \cdot d[\pi_2 \circ \mathtt{N}_f^{-1}](z, \Delta z) \\ = \ & \mathsf{Re}[i \frac{|f'(\pi_2(\mathtt{N}_f^{-1}(z)))|}{f'(\pi_2(\mathtt{N}_f^{-1}(z)))}[1 - i \frac{\zeta}{\det d\mathtt{N}_f(\mathtt{N}_f^{-1}(z))}].\Delta z], \quad \textit{and} \end{aligned}$$

(ii) $$d[\pi_2 \circ \mathtt{N}_f^{-1}](z, \Delta z) = \mathsf{Re}[\frac{|f'(\pi_2(\mathtt{N}_f^{-1}(z)))|}{f'(\pi_2(\mathtt{N}_f^{-1}(z)))} \frac{\Delta z}{\det d\mathtt{N}_f(\mathtt{N}_f^{-1}(z))}]$$

for $z \in \mathtt{A}_f$, $\Delta z \in \mathbb{C}$, and $\zeta \in \mathbb{R}$.

Proof. We have

$$\mathsf{Im}\, x \cdot \mathsf{Re}\, z - \mathsf{Re}\, x \cdot \mathsf{Im}\, z = \mathsf{Re}(i\, \bar{x} \cdot z) \qquad (\dagger)$$

for $x, z \in \mathbb{C}$ and

$$\begin{aligned} \overline{\partial_1 \mathtt{N}_f(\mathtt{N}_f^{-1}(z))} = \ & \overline{\partial_1 \mathtt{N}_f(\pi_1(\mathtt{N}_f^{-1}(z)), \pi_2(\mathtt{N}_f^{-1}(z)))} \\ \overset{6.2.1(i)}{=} \ & \overline{[-i \frac{f'(\pi_2(\mathtt{N}_f^{-1}(z)))}{|f'(\pi_2(\mathtt{N}_f^{-1}(z)))|}]} = \ i \frac{|f'(\pi_2(\mathtt{N}_f^{-1}(z)))|}{f'(\pi_2(\mathtt{N}_f^{-1}(z)))} \qquad (\dagger\dagger) \end{aligned}$$

for $z \in \mathbb{C}$. Now we compute

$$\begin{aligned} & \begin{pmatrix} d[\pi_1 \circ \mathtt{N}_f^{-1}](z, \Delta z) \\ d[\pi_2 \circ \mathtt{N}_f^{-1}](z, \Delta z) \end{pmatrix} \\ = \ & d[\mathtt{N}_f^{-1}](z) \begin{pmatrix} \mathsf{Re}\Delta z \\ \mathsf{Im}\Delta z \end{pmatrix} \qquad [\ \mathtt{N}_f^{-1} = \langle \pi_1 \circ \mathtt{N}_f^{-1}, \pi_2 \circ \mathtt{N}_f^{-1} \rangle\ ] \\ = \ & [d\mathtt{N}_f(\mathtt{N}_f^{-1}(z))]^{-1} \begin{pmatrix} \mathsf{Re}\Delta z \\ \mathsf{Im}\Delta z \end{pmatrix} \\ = \ & [\begin{pmatrix} \mathsf{Re}\, \partial_1 \mathtt{N}_f(\mathtt{N}_f^{-1}(z)) & \mathsf{Re}\, \partial_2 \mathtt{N}_f(\mathtt{N}_f^{-1}(z)) \\ \mathsf{Im}\, \partial_1 \mathtt{N}_f(\mathtt{N}_f^{-1}(z)) & \mathsf{Im}\, \partial_2 \mathtt{N}_f(\mathtt{N}_f^{-1}(z)) \end{pmatrix}]^{-1} \begin{pmatrix} \mathsf{Re}\Delta z \\ \mathsf{Im}\Delta z \end{pmatrix} \\ = \ & \frac{1}{\det d\mathtt{N}(\mathtt{N}_f^{-1}(z))} \begin{pmatrix} \mathsf{Im}\, \partial_2 \mathtt{N}_f(\mathtt{N}_f^{-1}(z)) & -\mathsf{Re}\, \partial_2 \mathtt{N}_f(\mathtt{N}_f^{-1}(z)) \\ -\mathsf{Im}\, \partial_1 \mathtt{N}_f(\mathtt{N}_f^{-1}(z)) & \mathsf{Re}\, \partial_1 \mathtt{N}_f(\mathtt{N}_f^{-1}(z)) \end{pmatrix} \begin{pmatrix} \mathsf{Re}\Delta z \\ \mathsf{Im}\Delta z \end{pmatrix} \\ \overset{(\dagger)}{=} \ & \frac{1}{\det d\mathtt{N}(\mathtt{N}_f^{-1}(z))} \begin{pmatrix} \mathsf{Re}(i \cdot \overline{\partial_2 \mathtt{N}_f(\mathtt{N}_f^{-1}(z))}.\Delta z) \\ -\mathsf{Re}(i \cdot \overline{\partial_1 \mathtt{N}_f(\mathtt{N}_f^{-1}(z))}.\Delta z) \end{pmatrix} \\ \overset{6.2.2(iii)}{=} \ & \begin{pmatrix} \mathsf{Re}(\overline{\partial_1 \mathtt{N}_f(\mathtt{N}_f^{-1}(z))}.\Delta z) \\ -\mathsf{Re}(i \cdot \overline{\partial_1 \mathtt{N}_f(\mathtt{N}_f^{-1}(z))}.\Delta z) \cdot [\det d\mathtt{N}(\mathtt{N}_f^{-1}(z))]^{-1} \end{pmatrix} \\ \overset{(\dagger\dagger)}{=} \ & \begin{pmatrix} \mathsf{Re}(i[\frac{|f'(\pi_2 \circ \mathtt{N}_f^{-1}(z))|}{f'(\pi_2 \circ \mathtt{N}_f^{-1}(z))}] \cdot \Delta z) \\ -\mathsf{Re}[\frac{|f'(\pi_2 \circ \mathtt{N}_f^{-1}(z))|}{f'(\pi_2 \circ \mathtt{N}_f^{-1}(z))} \frac{\Delta z}{\det d\mathtt{N}_f(\mathtt{N}_f^{-1}(z))}] \end{pmatrix}. \end{aligned}$$

If we multiply the equation above with $(1, -\zeta)$ from the left side we get (i), and if we multiply with $(0, 1)$ we get (ii). $\square$

Example 6.2.5. If $f(t) = e^{it}$, then

(i) $\mathtt{N}_f(\xi, t) = e^{it}(1 + \xi)$,

(ii) $\partial_1 \mathtt{N}_f(\xi, t) = e^{it}$,

(iii) $\partial_2 \mathtt{N}_f(\xi, t) = ie^{it}(1 + \xi)$,

(iv) $\mathsf{det}\, d\mathtt{N}_f(\xi, t) = (1 + \xi)$,

(v) $\pi_1(\mathtt{N}_f^{-1}(z)) = |z| - 1 = e^{\mathsf{Re} \log z} - 1$,

(vi) $\pi_2(\mathtt{N}_f^{-1}(z)) = -i \log \frac{z}{|z|} = \mathsf{Im} \log z$,

(vii) $d[\pi_1 \circ \mathtt{N}_f^{-1}](z, \Delta z) = \frac{1}{|z|} \cdot \mathsf{Re}(\overline{z} \cdot \Delta z) = |z| \cdot \mathsf{Re}(\frac{\Delta z}{z})$, and

(viii) $d[\pi_2 \circ \mathtt{N}_f^{-1}](z, \Delta z) = \mathsf{Im} \frac{\Delta z}{z}$.

6.3 Connectedness of $\mathtt{V}^+$, $\mathtt{V}^E$ and $\mathtt{V}$

In this section, we will show that the subsets $\mathtt{V}^+, \mathtt{V}^E, \mathtt{V} \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ are pathwise connected.

Lemma 6.3.1 (Path-connectedness of $\mathtt{V}^+$)**.** *The subset* $\mathtt{V}^+ \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ *is path-connected.*

Proof. Let $f \in \mathtt{V}^+$ and $f(t) = e^{it}(1 + a_1 e^{it} + a_2 e^{2it} + \cdots)$ its Fourier series with $a_1, a_2, \ldots \in \mathbb{C}$. By Lemma 2.3.8, the function $F : \overline{\mathbb{D}} \to \mathbb{C}$ defined by the Taylor series

$$F(z) := z(1 + a_1 z + a_2 z^2 + \cdots)$$

is smooth. Therefore the function

$$H : [0, 1] \times \mathbb{S}^1 \to \mathbb{C}, \quad (\lambda, \theta) \mapsto H(\lambda, \theta) = \int_0^1 F'(\lambda\, t\, e^{i\theta}) e^{i\theta} dt$$

is also smooth. Consider the case $0 < \lambda \leq 1$. By Lemma 2.4.6, the function $F : \mathbb{D} \to \mathbb{C}$ is univalent. This implies that the function

$$\mathbb{D} \to \mathbb{C}, \quad z \mapsto F(\lambda \cdot z)$$

is univalent for $\lambda \in (0, 1]$.

Hence, the function

$$
\begin{aligned}
f_\lambda : \mathbb{S}^1 \to \mathbb{C}, \quad \theta \mapsto H(\lambda, \theta) &= \int_0^1 F'(\lambda\, t\, e^{i\theta}) e^{i\theta}\, dt \\
&= \int_0^\lambda \frac{1}{\lambda} F'(\xi\, e^{i\theta}) e^{i\theta}\, d\xi \\
&= \frac{1}{\lambda}[F(\lambda\, e^{i\theta}) - \underbrace{F(0)}_{=0}] \\
&= e^{i\theta}(1 + a_1 \lambda e^{i\theta} + a_2 \lambda^2 e^{2i\theta} + \cdots),
\end{aligned}
$$

belongs to $\mathtt{V}^+$. In particular, for $\lambda = 1$ we have $f_\lambda = f$. Moreover, if $\lambda = 0$, then

$$H(0, \theta) = F'(0) \cdot e^{i\theta} = e^{i\theta}.$$

We conclude there exists a smooth function $H : [0,1] \times \mathbb{S}^1 \to \mathbb{C}$ with $H(0,t) = e^{it}$ and $H(1,t) = f(t)$, and the function

$$f_\lambda : \mathbb{S}^1 \to \mathbb{C}, \quad t \mapsto H(\lambda, t)$$

is an element of $\mathtt{V}^+$ for $\lambda \in [0,1]$. This defines a continuous path

$$[0,1] \to \mathtt{V}^+, \quad \lambda \mapsto \{\theta \mapsto H(\lambda, \theta)\}$$

in $\mathtt{V}^+$ connecting $f \in \mathtt{V}^+$ with the exponential function $\mathbb{S}^1 \to \mathbb{C}$, $t \mapsto e^{it}$. We conclude that $\mathtt{V}^+$ is path-connected. □

Lemma 6.3.2 (Path-connectedness of $\mathtt{V}^E$)**.** *The subset* $\mathtt{V}^E \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ *is path-connected.*

Proof. Let $f \in \mathtt{V}^E$ with $f(t) = e^{it}(r + a_1 e^{it} + a_2 e^{2it} + \cdots)$. The function f can be connected with $f/r \in \mathtt{V}^+$ by a line in $\mathtt{V}^E$. This shows that $\mathtt{V}^E$ is path-connected, because $\mathtt{V}^+$ is path-connected due to Lemma 6.3.1. □

Lemma 6.3.3 (Path-connectedness of $\mathtt{V}$)**.** *The subset* $\mathtt{V} \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ *is path-connected.*

Proof. By Theorem 4.2.1, every element $f \in \mathtt{V}$ can by written as

$$f = g \circ \gamma$$

with $g \in \mathtt{V}^E$ and $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$. Taking into account that $\mathsf{Diff}^+\mathbb{S}^1$ is path-connected, that $\mathtt{V}^E$ is path-connected (Lemma 6.3.3), and that the composition map

$$(g, \gamma) \mapsto f = g \circ \gamma$$

is continuous (Lemma 5.2.12), we conclude that the subset $\mathtt{V} \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is path-connected. □

Corollary 6.3.4. *If $f \in \mathtt{V}$, then the winding number of f' is equal* 1.

Proof. By Lemma 6.3.3, the set $\mathtt{V}$ is path-connected. The winding number function

$$\mathtt{w} : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times) \to \mathbb{Z}, \quad f \mapsto \frac{1}{2\pi} \int_{\mathbb{S}^1} \frac{f'(t)}{f(t)} dt$$

is continuous, and by Lemma 5.2.10(i) the derivative map

$$\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad f \mapsto f'$$

is continuous. Hence, $f \mapsto \mathtt{w}(f')$ is constant on $\mathtt{V}$ because $\mathtt{V}$ is connected. Since the exponential map $\mathbb{S}^1 \to \mathbb{C}, \quad t \mapsto e^{it}$ lies in $\mathtt{V}$, and $\mathtt{w}(\frac{d}{dt} e^{it}) = 1$, we are done. $\square$

Chapter 7

Tameness of the Composition map

In this chapter, we will prove that the composition map

$$\mathtt{C} : \mathtt{V}^E \times \mathsf{Diff}^+\mathbb{S}^1 \to \mathtt{V}, \quad (f,\gamma) \mapsto f \circ \gamma$$

is a tame diffeomorphism. In Section 7.1, we reduce the problem to show that a particular function $\hat{\theta}_f$ (see Definition 7.1.4) is a tame diffeomorphism. In Section 7.2, we compute the derivative of $\hat{\theta}_f$. Section 7.3 provides techniques. In Section 7.4, we construct for every $f \in \mathtt{V}^E$ a neighborhood $\mathtt{O}_f^E$, which is sufficiently small, so that we can compute in Section 7.5 the inverse function $d\hat{\theta}_f^{-1}$ explicitly. Thus, we can show that $d\hat{\theta}_f^{-1}$ is smooth-tame. This is a sufficient prerequisite for the Inverse Function Theorem of Nash and Moser (5.2.15), which yields that $\hat{\theta}_f$ is a tame diffeomorphism. In Section 7.6, we put everything together and obtain the main result of this chapter that $\mathtt{C}$ is a tame diffeomorphism.

7.1 Construction of $\mathtt{C}^{-1}$ on an Open Neighborhood

For every $f \in \mathtt{V}$ let $\mathtt{U}_f$ be a sufficiently small f-neighborhood for the following purpose. In this section, we define something like a chart of $\mathtt{V}$ with coordinate spaces $\mathcal{C}^\infty(\mathbb{S}^1,\mathbb{R})$ and $\mathsf{Diff}^+\mathbb{S}^1$ by

$$\begin{aligned} \langle \hat{\Theta}_f, \hat{\Sigma}_f \rangle : \mathtt{U}_f &\to \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{R}) \times \mathsf{Diff}^+\mathbb{S}^1, \\ \eta &\mapsto < (\pi_1 \circ \mathtt{N}_f^{-1} \circ \eta) \circ (\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta)^{-1} \ , \ (\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta) > . \end{aligned}$$

Since $\mathtt{V} = \bigcup_{f\in\mathtt{V}} \mathtt{U}_f$, all these charts together are an atlas of $\mathtt{V}$. Furthermore, we will show that the restriction

$$\hat{\theta}_f : \mathtt{U}_f \cap \mathtt{V}^E \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$$

of $\hat{\Theta}_f$ is injective (Lemma 7.1.9) using that the composition map $\mathtt{C}$ is bijective. With these three functions $\hat{\Sigma}_f$, $\hat{\Theta}_f$, and $\hat{\theta}_f^{-1}$ we can express the inverse of the composition map $\mathtt{C}$ as

$$\begin{aligned}\mathtt{C}^{-1} &: \mathtt{U}_f \to \mathtt{V}^E \times \mathsf{Diff}^+\mathbb{S}^1,\\ &\eta \mapsto ([\hat{\theta}_f]^{-1}(\hat{\Theta}_f(\eta)) \quad , \quad [\hat{\Sigma}_f([\hat{\theta}_f]^{-1}(\hat{\Theta}_f(\eta)))]^{-1} \circ [\hat{\Sigma}_f(\eta)]).\end{aligned}$$

The function $\hat{\Sigma}_f$, and $\hat{\Theta}_f$ are also smooth-tame by construction. If we could show that $\hat{\theta}_f$ is smooth-tame, we would be finished. This remaining problem will be solved in the other sections of this chapter.

Definition 7.1.1. For $f \in \mathtt{V}$ we define the subset

$$\mathtt{U}_f = \{\, \eta \in \mathtt{V} \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \,:\, \forall_{t\in\mathbb{S}^1}\ \eta(t) \in \mathtt{A}_f \wedge (\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta)'(t) \neq 0 \,\}.$$

Remark: Recall from Lemma 6.2.3 that

$$\mathtt{N}_f : (-\mathtt{r}_f, \mathtt{r}_f) \times \mathbb{S}^1 \to \mathtt{A}_f, \quad (\xi, t) \mapsto \mathtt{N}_f(\xi, t)$$

is a diffeomorphism for $f \in \mathtt{V}$ with the tube radius $\mathtt{r}_f$ defined in 6.1.1.

Lemma 7.1.2. *If $\eta \in \mathtt{U}_f$ and $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$, then $\eta \circ \gamma \in \mathtt{U}_f$ and $\hat{\Lambda}(\eta) \in \mathtt{U}_f$.*

Proof. If we take $\gamma'(t) \neq 0$ into account, we get

$$(\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta \circ \gamma)'(t) \neq 0$$

and therefore $\eta \circ \gamma \in \mathtt{U}_f$. By Corollary 4.2.3 there exists a diffeomorphism $\hat{\Gamma}(\eta) \in \mathsf{Diff}^+\mathbb{S}^1$ such that $\hat{\Lambda}(\eta) = \eta \circ \hat{\Gamma}(\eta)^{-1}$, and therefore $\hat{\Lambda}(\eta) \in \mathtt{U}_f$. □

Definition 7.1.3. Let $f \in \mathtt{V}$. We define two functions:

(i) $\hat{\Theta}_f : \mathtt{U}_f \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$ by $\hat{\Theta}_f(\eta) = (\pi_1 \circ \mathtt{N}_f^{-1} \circ \eta) \circ (\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta)^{-1}$.

(ii) $\hat{\Sigma}_f : \mathtt{U}_f \to \mathsf{Diff}^+\mathbb{S}^1$ by $\hat{\Sigma}_f(\eta) = (\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta)$.

Remark: The term $(\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta)^{-1}$ in the definition of $\hat{\Theta}_f$ guarantees the invariance, i.e., $\hat{\Theta}_f(\eta) = \hat{\Theta}_f(\eta \circ \gamma)$ for all $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$.

Definition 7.1.4. For $f \in \mathtt{V}$ we define

$$\hat{\theta}_f : \mathtt{V}^E \cap \mathtt{U}_f \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$$

to be the restriction of $\hat{\Theta}_f$ to $\mathtt{V}^E \cap \mathtt{U}_f$.

In the rest of this section, we express the functions $\hat{\Lambda}, \hat{\Gamma}$ (restricted to $\mathtt{U}_f$) in terms of $\hat{\Theta}_f$, $\hat{\Sigma}_f$, and $\hat{\theta}_f^{-1}$. The fact that $\hat{\theta}_f$ is injective will be shown in Lemma 7.1.9 using the result of Chapter 4 that the composition map $\mathtt{C}$ is bijective.

Lemma 7.1.5. *If $f \in \mathtt{V}$, then $\mathtt{U}_f \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is an open neighborhood of f.*

Proof. 1. Step: $\mathtt{U}_f \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is open:
By Lemma 6.1.8, the subset $\mathtt{V} \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is open. Since $\mathtt{A}_f \subseteq \mathbb{C}$ is open by Lemma 6.2.3 the subset

$$\mathcal{C}^\infty(\mathbb{S}^1, \mathtt{A}_f) \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$$

is an open subset by Lemma 5.2.9. The right composition

$$\mathsf{R}_{\pi_2 \circ \mathtt{N}_f^{-1}} : \mathcal{C}^\infty(\mathbb{S}^1, \mathtt{A}_f) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$$

is smooth by Lemma 5.2.11, because $\pi_2 \circ \mathtt{N}_f^{-1} : \mathtt{A}_f \to \mathbb{C}$ is smooth. By Lemma 5.2.10, the derivative map

$$D : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \lambda \mapsto \lambda'$$

is continuous, and therefore the composition

$$D \circ \mathsf{R}_{\pi_2 \circ \mathtt{N}_f^{-1}} : \mathcal{C}^\infty(\mathbb{S}^1, \mathtt{A}_f) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$$

is continuous. Moreover, $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times) \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is an open subset by Lemma 5.2.9. The preimage

$$\begin{aligned} W : &= (D \circ \mathsf{R}_{\pi_2 \circ \mathtt{N}_f^{-1}})^{-1})(\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times) \\ &= \{\, \eta \in \mathcal{C}^\infty(\mathbb{S}^1, \mathtt{A}_f) \,:\, \forall_{t \in \mathbb{S}^1}\ (\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta)'(t) \neq 0 \,\} \end{aligned}$$

is an open subset of $\mathcal{C}^\infty(\mathbb{S}^1, \mathtt{A}_f)$, and we conclude that

$$U_f = \mathtt{V} \cap W$$

is open in $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$.
2. Step: $f \in \mathtt{U}_f$:
Definition 6.1.3 implies $f \in \mathtt{A}_f$ for all $t \in \mathbb{S}^1$. Futhermore, Lemma 6.1.10 yields that $\pi_2 \circ \mathtt{N}_f^{-1} \circ f = \mathsf{id}_{\mathbb{S}^1}$ and therefore $(\pi_2 \circ \mathtt{N}_f^{-1} \circ f)' \equiv 1 \neq 0$. This shows $f \in \mathtt{U}_f$. □

Lemma 7.1.6. *For $f \in \mathtt{V}$ the two maps*

(i) $\hat{\Sigma}_f : \mathtt{U}_f \to \mathsf{Diff}^+\mathbb{S}^1$, *and*

(ii) $\hat{\Theta}_f : \mathtt{U}_f \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$

are smooth-tame.

Proof. (i):
The functions $\pi_2 \circ \mathtt{N}_f^{-1} : \mathtt{A}_f \to \mathbb{S}^1$ and $\pi_1 \circ \mathtt{N}_f^{-1} : \mathtt{A}_f \to \mathbb{R}$ are smooth, and therefore the left translations

$$\begin{aligned} \mathsf{L}_{\pi_2 \circ \mathtt{N}_f^{-1}} &: \mathcal{C}^\infty(\mathbb{S}^1, \mathtt{A}_f) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{S}^1), \qquad \eta \mapsto \pi_2 \circ \mathtt{N}_f^{-1} \circ \eta \\ \mathsf{L}_{\pi_1 \circ \mathtt{N}_f^{-1}} &: \mathcal{C}^\infty(\mathbb{S}^1, \mathtt{A}_f) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}), \qquad \eta \mapsto \pi_1 \circ \mathtt{N}_f^{-1} \circ \eta \end{aligned}$$

are smooth-tame by Lemma 5.3.5. Since we have $\mathtt{U}_f \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathtt{A}_f)$ (Definition 7.1.1), $\mathsf{Diff}^+\mathbb{S}^1 \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{S}^1)$, and $\hat{\Sigma}_f = \mathsf{L}_{\pi_2 \circ \mathtt{N}_f^{-1}}$, the part (i) of this lemma is proved.
(ii):
By Lemma 5.2.13(ii), the group inversion

$$\iota : \mathsf{Diff}^+\mathbb{S}^1 \to \mathsf{Diff}^+\mathbb{S}^1, \quad \gamma \mapsto \gamma^{-1}$$

is smooth-tame and by Lemma 5.2.12 the composition map

$$\mathtt{C} : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}) \times \mathsf{Diff}^+\mathbb{S}^1 \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$$

is also smooth-tame. From this we obtain that the map

$$\hat{\Theta}_f : \mathtt{U}_f \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathtt{A}_f) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$$

with

$$\hat{\Theta}_f(\eta) = (\pi_1 \circ \mathtt{N}_f^{-1} \circ \eta) \circ (\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta)^{-1} = C(\mathsf{L}_{\pi_1 \circ \mathtt{N}_f^{-1}}(\eta), \iota(\mathsf{L}_{\pi_2 \circ \mathtt{N}_f^{-1}}(\eta)))$$

is smooth-tame. The statement (ii) is proved. $\square$

Lemma 7.1.7. *Given $f \in \mathtt{V}$. If $\eta_1, \eta_2 \in \mathtt{U}_f$, then the following two statements are equivalent:*

(i) $\eta_1 \circ [\hat{\Sigma}_f(\eta_1)]^{-1} = \eta_2 \circ [\hat{\Sigma}_f(\eta_2)]^{-1}$.

(ii) $\hat{\Theta}_f(\eta_1) = \hat{\Theta}_f(\eta_2)$.

Proof. (i)$\Longrightarrow$(ii):

$$\begin{aligned} \hat{\Theta}_f(\eta_1) &\overset{7.1.3}{=} \pi_2 \circ \mathtt{N}_f^{-1} \circ \eta_1 \circ [\hat{\Sigma}_f(\eta_1)]^{-1} \overset{(i)}{=} \pi_2 \circ \mathtt{N}_f^{-1} \circ \eta_2 \circ [\hat{\Sigma}_f(\eta_2)]^{-1} \\ &\overset{7.1.3}{=} \hat{\Theta}_f(\eta_2). \end{aligned}$$

(ii)$\Longrightarrow$(i): The equation $\hat{\Theta}_f(\eta_1) = \hat{\Theta}_f(\eta_2)$ is equivalent to

$$(\pi_1 \circ \mathtt{N}_f^{-1} \circ \eta_1) \circ [\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta_1]^{-1} = (\pi_1 \circ \mathtt{N}_f^{-1} \circ \eta_2) \circ [\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta_2]^{-1}$$

according to the Definition 7.1.3(i) of $\hat{\Theta}_f$. Moreover,

$$\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta_1 \circ [\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta_1]^{-1} = \mathsf{id}_{\mathbb{S}^1} = \pi_2 \circ \mathtt{N}_f^{-1} \circ \eta_2 \circ [\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta_2]^{-1}$$

and therefore

$$\mathtt{N}_f^{-1} \circ \eta_1 \circ [\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta_1]^{-1} = \mathtt{N}_f^{-1} \circ \eta_2 \circ [\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta_2]^{-1}.$$

The composition of the bijective function

$$\mathtt{N}_f : (-\mathtt{r}_f, \mathtt{r}_f) \times \mathbb{S}^1 \to \mathtt{A}_f$$

from the left-hand, to to this equation yields

$$\eta_1 \circ [\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta_1]^{-1} = \eta_2 \circ [\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta_2]^{-1}.$$

The last equation is equivalent to

$$\eta_1 \circ [\hat{\Sigma}_f(\eta_1)]^{-1} = \eta_2 \circ [\hat{\Sigma}_f(\eta_2)]^{-1}$$

by the Definition 7.1.3(ii) of $\hat{\Sigma}_f$. □

Lemma 7.1.8. *If $f \in \mathtt{V}$, then*
(i) $\hat{\Sigma}_f(f) = \mathsf{id}_{\mathbb{S}^1}$,
(ii) $\hat{\Sigma}_f(\hat{\Lambda}(f)) = [\hat{\Gamma}(f)]^{-1}$,
(iii) $\hat{\Theta}_f(f) = 0$, *and*
(iv) $\hat{\Theta}_f(\hat{\Lambda}(f)) = 0$.

Proof. (i): $\hat{\Sigma}_f(f)(t) \overset{7.1.3(ii)}{=} (\pi_2 \circ \mathtt{N}_f^{-1} \circ f)(t) \overset{6.1.10(ii)}{=} t$.
(ii):

$$\begin{aligned}\hat{\Sigma}_f(f)(\hat{\Lambda}(f)(t)) &\overset{7.1.3(ii)}{=} (\pi_2 \circ \mathtt{N}_f^{-1} \circ \hat{\Lambda}(f))(t) \\ &\overset{4.2.3(i)}{=} (\pi_2 \circ \mathtt{N}_f^{-1} \circ f \circ \hat{\Gamma}(f)^{-1})(t) \\ &\overset{6.1.10(ii)}{=} \hat{\Gamma}(f)^{-1}(t).\end{aligned}$$

(iii): $\hat{\Theta}_f(f) \overset{7.1.3(i)}{=} (\pi_1 \circ \mathtt{N}_f^{-1} \circ f) \circ (\pi_2 \circ \mathtt{N}_f^{-1} \circ f)^{-1} \overset{6.1.10(i)}{=} 0$.
(iv): $\hat{\Theta}_f(\hat{\Lambda}(f)) \overset{7.1.3(i)}{=} \hat{\Theta}_f(\hat{\Lambda}(f) \circ \hat{\Gamma}(f)) \overset{4.2.3(i)}{=} \hat{\Theta}_f(f) \overset{(iii)}{=} 0$. □

Lemma 7.1.9. *If $f \in \mathtt{V}$, then the map $\hat{\theta}_f : \mathtt{U}_f \cap \mathtt{V}^E \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$ is injective.*

Proof. Recall (7.1.4), $\hat{\theta}_f$ is the restriction of $\hat{\Theta}_f$ to $\mathtt{U}_f \cap \mathtt{V}^E$. Let $\eta_1, \eta_2 \in \mathtt{V}^E \cap \mathtt{U}_f$ and assume

$$\hat{\Theta}_f(\eta_1) = \hat{\Theta}_f(\eta_2).$$

The assumption implies

$$\eta_1 \circ [\hat{\Sigma}_f(\eta_1)]^{-1} = \eta_2 \circ [\hat{\Sigma}_f(\eta_2)]^{-1}$$

using Lemma 7.1.7(i). Since $\eta_1, \eta_2 \in \mathtt{V}^E$, we conlude $\eta_1 = \eta_2$ by Theorem 4.2.1. Hence, $\hat{\theta}_f = \hat{\Theta}_f|_{\mathtt{U}_f \cap \mathtt{V}^E}$ is injective. □

Proposition 7.1.10. *If $f \in \mathtt{V}$, then*
(i) $\hat{\Lambda}(\eta) = [\hat{\theta}_f]^{-1}(\hat{\Theta}_f(\eta))$ *for all $\eta \in \mathtt{U}_f$, and*
(ii) $\hat{\Gamma}(\eta) = [\hat{\Sigma}_f([\hat{\theta}_f]^{-1}(\hat{\Theta}_f(\eta)))]^{-1} \circ [\hat{\Sigma}_f(\eta)]$ *for all $\eta \in \mathtt{U}_f$.*

Proof. (i): The computation

$$\begin{aligned}
\hat{\Theta}_f(\eta) &\overset{4.2.3(i)}{=} \hat{\Theta}_f(\hat{\Lambda}(\eta) \circ \hat{\Gamma}(\eta)) \\
&\overset{7.1.3(i)}{=} (\pi_1 \circ \mathtt{N}_f^{-1} \circ \hat{\Lambda}(\eta) \circ \hat{\Gamma}(\eta)) \circ (\pi_2 \circ \mathtt{N}_f^{-1} \circ \hat{\Lambda}(\eta) \circ \hat{\Gamma}(\eta))^{-1} \\
&= (\pi_1 \circ \mathtt{N}_f^{-1} \circ \hat{\Lambda}(\eta)) \circ (\pi_2 \circ \mathtt{N}_f^{-1} \circ \hat{\Lambda}(\eta))^{-1} \\
&\overset{7.1.3(i)}{=} \hat{\Theta}_f(\hat{\Lambda}(\eta)) \\
&\overset{7.1.4}{=} \hat{\theta}_f(\hat{\Lambda}(\eta))
\end{aligned}$$

for $\eta \in \mathtt{U}_f$ and the injectivity of $\hat{\theta}_f$ yield the statement.
(ii):

$$\begin{aligned}
\hat{\Gamma}(\eta) &= [\hat{\Sigma}_f(\hat{\Lambda}(\eta))]^{-1} \circ [\hat{\Sigma}_f(\hat{\Lambda}(\eta))] \circ \hat{\Gamma}(\eta) \\
&\overset{7.1.3(ii)}{=} [\hat{\Sigma}_f(\hat{\Lambda}(\eta))]^{-1} \circ \pi_2 \circ \mathtt{N}_f^{-1} \circ \hat{\Lambda}(\eta) \circ \hat{\Gamma}(\eta) \\
&\overset{4.2.3(i)}{=} [\hat{\Sigma}_f(\hat{\Lambda}(\eta))]^{-1} \circ \pi_2 \circ \mathtt{N}_f^{-1} \circ \eta \\
&\overset{7.1.3(ii)}{=} [\hat{\Sigma}_f(\hat{\Lambda}(\eta))]^{-1} \circ \hat{\Sigma}_f(\eta) \\
&= [\hat{\Sigma}_f([\hat{\theta}_f]^{-1}(\hat{\Theta}_f(\eta)))]^{-1} \circ \hat{\Sigma}_f(\eta).
\end{aligned}$$

□

In 7.1.6, we proved that $\hat{\Theta}_f$ and $\hat{\Sigma}_f$ are smooth-tame. If we could show that $[\hat{\theta}_f]^{-1}$ is also smooth-tame, we would follow that $\hat{\Lambda}$ and $\hat{\Gamma}$ are smooth-tame, and hence that

$$\mathtt{C} : \mathtt{V}^E \times \mathsf{Diff}^+\mathbb{S}^1 \to \mathtt{V}$$

is a tame diffeomorphism. In the following sections, we will show that the derivative of $\hat{\theta}_f$ is invertible (with respect to right argument) and tame, so that we can apply Nash-Moser Theorem to prove that $\hat{\theta}_f$ is a tame diffeomorphism and $\hat{\theta}_f^{-1}$ is smooth tame.

7.2 Derivative of $\hat{\theta}_f$

The remaining task of this chapter is to show that $\hat{\theta}_f^{-1}$ is smooth-tame. We will show this with the Inverse Function Theorem of Nash and Moser. To do so, we must show that the derivative of $\hat{\theta}_f : \mathtt{U}_f \cap \mathtt{V}^E \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$ with respect to the right argument is invertible[1] and this inverse is smooth-tame. In this section, we compute the derivative of $\hat{\theta}_f$. More precisely, we compute the derivative of $\hat{\Theta}_f$, which can be restricted.

Definition 7.2.1. We define $\mathtt{E}_1 \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ to be the subset of functions with Fourier series of the form $f(t) = e^{it}(r + a_1 e^{it} + a_2 e^{2it} + \cdots)$ with $r \in \mathbb{R}$ and $a_1, a_2, \ldots \in \mathbb{C}$.

Lemma 7.2.2. *The subset $\mathtt{V}^E \subseteq \mathtt{E}_1$ of the real Fréchet space $\mathtt{E}_1$ is open.*

Proof. We have

$$\mathtt{V}^E = \{\, f \in \mathtt{E}_1 \;:\; f \in \mathtt{V} \text{ and } \int_{\mathbb{S}^1} e^{it} f(t)dt > 0 \,\}.$$

Since the mapping

$$\alpha : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathbb{C}, \quad f \mapsto \int_{\mathbb{S}^1} e^{it} f(t)dt$$

is continuous, the subset $\alpha^{-1}(\mathbb{R}^+) \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is open. Moreover, by Lemma 6.1.8 the subset $\mathtt{V} \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is also open. Hence, we conclude that

$$\mathtt{V}^E = \alpha^{-1}(\mathbb{R}^+) \cap \mathtt{V} \cap \mathtt{E}_1$$

is an open subset of the vector space $\mathtt{E}_1$. $\square$

[1] We will actually show this for a smaller neighborhood of $f \in \mathtt{V}^E$.

Remark: We can consider $\mathtt{E}_1$ as the tangent space of $\mathtt{V}^E$ and therefore of $\mathtt{V}^E \cap \mathtt{U}_f$. In particular, the derivative of $\hat{\theta}_f$ has the form

$$d\hat{\theta}_f : \mathtt{V}^E \cap \mathtt{U}_f \times \mathtt{E}_1 \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}).$$

Definition 7.2.3. We define $\mathtt{W} \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ to be the subset of all smooth functions $f : \mathbb{S}^1 \to \mathbb{C}$ with $f(t) \neq 0$ for $t \in \mathbb{S}^1$ and winding number

$$\mathtt{w}(f) = \int_{\mathbb{S}^1} \frac{f'(t)}{f(t)} dt = 0.$$

Lemma 7.2.4. *The subset* $\mathtt{W} \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ *is open.*

Proof. By Lemma 5.2.9, the subset $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times) \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is open, and $\mathtt{w}^{-1}(0)$ is open by Lemma 5.3.7. $\square$

Definition 7.2.5. Let $f \in \mathtt{V}$ and $\eta \in \mathtt{U}_f$. We define

$$\hat{k}_f(\eta) : \mathbb{S}^1 \to \mathbb{C}, \quad \tau \mapsto \hat{k}_f(\eta)(\tau)$$

by

$$\hat{k}_f(\eta)(\tau) := i\frac{|f'(\pi_2(\mathtt{N}_f^{-1}(\eta(\tau))))|}{f'(\pi_2(\mathtt{N}_f^{-1}(\eta(\tau))))} \cdot \left[1 - i\frac{[\hat{\Theta}_f(\eta)]'(\hat{\Sigma}_f(\eta)(\tau))}{\mathsf{det}\, d\mathtt{N}_f(\mathtt{N}_f^{-1}(\eta(\tau)))}\right] \cdot e^{i\tau}.$$

Remark:
Since $\eta \in \mathtt{U}_f$, we have $\eta(t) \in \mathtt{A}_f$ and $\mathsf{det}\, d\mathtt{N}_f(\mathtt{N}_f^{-1}(\eta(\tau))) > 0$ due to Lemma 6.2.2(ii).

Lemma 7.2.6. *Let* $f \in \mathtt{V}$. *Then the derivative of*

$$\hat{\Theta}_f : \mathtt{U}_f \mapsto \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$$

reads

$$d\hat{\Theta}_f : \mathtt{U}_f \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$$

with

$$d\hat{\Theta}_f(\eta, \Delta\eta)(t) = \mathsf{Re}[\hat{k}_f(\eta)(\tau) \cdot e^{-i\tau}\Delta\eta(\tau)]|_{\tau=[\hat{\Sigma}_f(\eta)]^{-1}(t)}.$$

Proof. By Definition 7.1.3, we have

$$\hat{\Theta}_f(\eta) = (\pi_1 \circ \mathtt{N}_f^{-1} \circ \eta) \circ (\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta)^{-1} = \mathsf{L}_{\pi_1 \circ \mathtt{N}_f^{-1}}(\eta) \circ \iota(\hat{\Sigma}_f(\eta))$$

and get

$$\begin{aligned}
& d\hat{\Theta}_f(\eta, \Delta\eta)(t) \\
\overset{5.5.5}{=} \;& d[\mathsf{L}_{\pi_1 \circ \mathtt{N}_f^{-1}}](\eta, \Delta\eta)([\hat{\Sigma}_f(\eta)]^{-1}(t)) \\
& \qquad - d[\hat{\Theta}_f(\eta)]_t . d[\underbrace{\hat{\Sigma}_f}_{\mathsf{L}_{\pi_2 \circ \mathtt{N}_f^{-1}}}](\eta, \Delta\eta)([\hat{\Sigma}_f(\eta)]^{-1}(t)) \\
\overset{5.5.1(i)}{=} \;& d[\pi_1 \circ \mathtt{N}_f^{-1}](\eta(\tau), \Delta\eta(\tau)\,) \\
& \qquad - [\hat{\Theta}_f(\eta)]'(\hat{\Sigma}_f(\eta)(\tau)) \cdot d[\pi_2 \circ \mathtt{N}_f^{-1}](\eta(\tau), \Delta\eta(\tau))|_{\tau = [\hat{\Sigma}_f(\eta)]^{-1}(t)} \\
\overset{6.2.4(i)}{=} \;& \mathsf{Re}[i \frac{|f'(\pi_2(\mathtt{N}_f^{-1}(\eta(\tau))))|}{f'(\pi_2(\mathtt{N}_f^{-1}(\eta(\tau))))}[1 - i \frac{[\hat{\Theta}_f(\eta)]'(\hat{\Sigma}_f(\eta)(\tau))}{\det d\mathtt{N}_f(\mathtt{N}_f^{-1}(\eta(\tau)))}].\Delta\eta(\tau)]|_{\tau = [\hat{\Sigma}_f(\eta)]^{-1}(t)} \\
\overset{7.2.5}{=} \;& \mathsf{Re}[\hat{k}_f(\eta)(\tau) \cdot e^{-i\tau} \Delta\eta(\tau)]|_{\tau = [\hat{\Sigma}_f(\eta)]^{-1}(t)}.
\end{aligned}$$

□

Corollary 7.2.7. *The derivative of*

$$\hat{\theta}_f : \mathtt{U}_f \cap \mathtt{V}^E \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$$

is

$$d\hat{\theta}_f : \mathtt{U}_f \cap \mathtt{V}^E \times \mathtt{E}_1 \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}), \quad (\eta, \Delta\eta) \mapsto d\hat{\theta}_f(\eta, \Delta\eta)$$

with

$$d\hat{\theta}_f(\eta, \Delta\eta)(t) = \mathsf{Re}[\hat{k}_f(\eta)(\tau) \cdot e^{-i\tau} \Delta\eta(\tau)]|_{\tau = [\hat{\Sigma}_f(\eta)]^{-1}(t)}.$$

In the rest of this section, we discuss some properties of the function $\hat{k}_f$, which plays a central role in the following sections.

Lemma 7.2.8. *If $f \in \mathtt{V}$, then*

(i) $\hat{k}_f(f \circ \gamma)(\tau) = i \frac{|(f \circ \gamma)'(\tau)|}{(f \circ \gamma)'(\tau)} \cdot e^{i\tau}$ *for all* $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$*, and*

(ii) $\hat{k}_f(\hat{\Lambda}(f))(\tau) = i \frac{|\hat{\Lambda}(f)'(\tau)|}{\hat{\Lambda}(f)'(\tau)} e^{i\tau}.$

Proof. (i):

$$\hat{\Theta}_f(f\circ\gamma) \overset{7.1.3(i)}{=} (\pi_1\circ \mathtt{N}_f^{-1}\circ f\circ\gamma)\circ(\pi_2\circ\mathtt{N}_f^{-1}\circ f\circ\gamma)^{-1} \overset{6.1.10(i)}{=} 0$$

implies

$$[\hat{\Theta}_f(f\circ\gamma)]' \equiv 0, \qquad (\dagger)$$

and we compute

$$\begin{aligned}
&\hat{k}_f(f\circ\gamma)(\tau)\\
\overset{7.2.5}{=}\ & i\frac{|f'(\pi_2(\mathtt{N}_f^{-1}(f(\gamma(\tau)))))|}{f'(\pi_2(\mathtt{N}_f^{-1}(f(\gamma(\tau)))))}\cdot\left[1-i\frac{[\hat{\Theta}_f(f\circ\gamma)]'(\hat{\Sigma}_f(f\circ\gamma)(\tau))}{\det d\mathtt{N}_f(\mathtt{N}_f^{-1}(f(\gamma(\tau))))}\right]\cdot e^{i\tau}\\
\overset{(\dagger)}{=}\ & i\frac{|f'(\pi_2(\mathtt{N}_f^{-1}(f(\gamma(\tau)))))|}{f'(\pi_2(\mathtt{N}_f^{-1}(f(\gamma(\tau)))))}\cdot e^{i\tau}\\
\overset{6.1.10(ii)}{=}\ & i\frac{|f'(\gamma(\tau))|}{f'(\gamma(\tau))}\cdot e^{i\tau}\\
=\ & i\frac{|f'(\gamma(\tau))\cdot\gamma'(\tau)|}{f'(\gamma(\tau))\cdot\gamma'(\tau)}\cdot e^{i\tau}\\
=\ & i\frac{|(f\circ\gamma)'(\tau)|}{(f\circ\gamma)'(\tau)}\cdot e^{i\tau}.
\end{aligned}$$

(ii): Using Corollary 4.2.3(i)

$$f = \hat{\Lambda}(f)\circ\hat{\Gamma}(f)$$

and substituting $\gamma = \hat{\Gamma}(f)^{-1}$ into part (i) yield the statement. ☐

Lemma 7.2.9. *If $f\in\mathtt{V}$ and $\eta\in\mathtt{U}_f$, then $\hat{k}_f(\eta)\in\mathtt{W}$. In particular, we have $\hat{k}_f(\eta)(t)\neq 0$ for all $t\in\mathbb{S}^1$.*

Proof. Let $\gamma(\tau) := \pi_2(\mathtt{N}_f^{-1}(\eta(\tau)))$ for all $\tau\in\mathbb{S}^1$. So, by Definition 7.2.5 we have

$$\begin{aligned}
\hat{k}_f(\eta)(\tau) &:= i\frac{|f'(\gamma(\tau))|}{f'(\gamma(\tau))}\cdot\left[1-i\frac{[\hat{\Theta}_f(\eta)]'(\hat{\Sigma}_f(\eta)(\tau))}{\det d\mathtt{N}_f(\mathtt{N}_f^{-1}(\eta(\tau)))}\right]\cdot e^{i\tau}\\
&= h_1(\tau)\cdot h_2(\tau)\cdot h_3(\tau)\cdot h_4(\tau)
\end{aligned}$$

with

$$\begin{aligned} h_1(\tau) &:= |f'(\gamma(\tau))| \\ h_2(\tau) &:= \frac{1}{f'(\gamma(\tau))} \\ h_3(\tau) &:= 1 - i\frac{[\hat{\Theta}_f(\eta)]'(\hat{\Sigma}_f(\eta)(\tau))}{\det d\mathtt{N}_f(\mathtt{N}_f^{-1}(\eta(\tau)))} \\ h_4(\tau) &:= e^{i\tau}. \end{aligned}$$

We have $\mathtt{w}(h_1) = 0$, because $h_1(\tau) \in \mathbb{R}^+$, and we have

$$\mathtt{w}(h_2) = \mathtt{w}(\frac{1}{f' \circ \gamma}) \overset{2.4.2(ii)}{=} -\mathtt{w}(f' \circ \gamma) \overset{2.4.2(iv)}{=} -\mathtt{w}(f') \overset{6.3.4}{=} -1.$$

Since the image of h_3 is a line in the complex plane, which is parallel to the y-axes, $\mathtt{w}(h_3) = 0$. And finally, $\mathtt{w}(h_4) = 1$. Hence, we conclude

$$\mathtt{w}(\hat{k}_f(\eta)) = \mathtt{w}(h_1 \cdot h_2 \cdot h_3 \cdot h_4) = \mathtt{w}(h_1) + \mathtt{w}(h_2) + \mathtt{w}(h_3) + \mathtt{w}(h_4) = 0.$$

□

Lemma 7.2.10. *For $f \in \mathtt{V}$ the map*

$$\hat{k}_f : \mathtt{U}_f \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathtt{A}_f) \to \mathtt{W} \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \eta \mapsto \hat{k}_f(\eta)$$

is smooth-tame.

Proof. 1. Step:
The maps $\hat{\Theta}_f$ and $\hat{\Sigma}_f$ are smooth-tame by Lemma 7.1.6. The derivative map is smooth-tame by Lemma 5.2.10(i). The composition map is smooth-tame by Lemma 5.2.12. We conclude that

$$\chi_f : \mathtt{U}_f \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}), \quad \eta \mapsto \chi_f(\eta) = [\hat{\Theta}_f(\eta)]' \circ \hat{\Sigma}_f(\eta)$$

is a smooth-tame map.
2. Step:
Define a smooth map $q_f : \mathtt{A}_f \times \mathbb{R} \to \mathbb{C}$ by

$$q_f(z, \zeta) := i\frac{|f'(\pi_2(\mathtt{N}_f^{-1}(z)))|}{f'(\pi_2(\mathtt{N}_f^{-1}(z)))} \cdot \left[1 - i\frac{\zeta}{\det d\mathtt{N}_f(\mathtt{N}_f^{-1}(z))}\right].$$

The left composition

$$\mathsf{L}_{q_f} : \mathcal{C}^\infty(\mathbb{S}^1, \mathtt{A}_f) \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$$

is smooth-tame by Lemma 5.3.5, if we identify $\mathcal{C}^\infty(\mathbb{S}^1, \mathtt{A}_f) \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$ with $\mathcal{C}^\infty(\mathbb{S}^1, \mathtt{A}_f \times \mathbb{R})$.
3. Step:
The map

$$\mathtt{U}_f \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathtt{A}_f) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \eta \mapsto \mathsf{L}_{q_f}(\eta, \chi_f(\eta))$$

is smooth-tame by Lemma 5.2.8. Moreover,

$$\begin{aligned}
&\mathsf{L}_{q_f}(\eta, \chi_f(\eta))(\tau) \cdot e^{i\tau} \\
=\ & q_f(\eta(\tau), \chi_f(\eta)(\tau)) \cdot e^{i\tau} \\
=\ & i\frac{|f'(\pi_2(\mathtt{N}_f^{-1}(\eta(\tau))))|}{f'(\pi_2(\mathtt{N}_f^{-1}(\eta(\tau))))} \cdot \left[1 - i\frac{\chi_f(\eta)(\tau)}{\mathsf{det}\, d\mathtt{N}_f(\mathtt{N}_f^{-1}(\eta(\tau)))}\right] \cdot e^{i\tau} \\
=\ & \hat{k}_f(\eta)(\tau).
\end{aligned}$$

Since the multiplication with $e^{i\tau}$ is linear-tame, we conclude that $\hat{k}_f$ is smooth-tame. □

7.3 Calculus of Linear Projections

This section provides us with technical stuff used in several ways, particularly in Section 7.5 for inverting $d\hat{\theta}_f$. We define several linear mappings on $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ and prove relations between them. All these mappings respect the decomposition

$$\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) = V^0 \oplus \bigoplus_{n=1}^{\infty} V^n$$

with $V^n = \{\, f \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \;:\; f(t) = be^{int} + ce^{-int} \text{ with } b, c \in \mathbb{C} \,\}$ and V^0 the subset of constant functions.

Definition 7.3.1. Consider the real algebra

$$\mathcal{A} := \left\{ \left(\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array} \right) \in \mathsf{Mat}(6 \times 6, \mathbb{R}) \;:\; B \in \mathsf{Mat}(2 \times 2, \mathbb{R}), C \in \mathsf{Mat}(4 \times 4, \mathbb{R}) \right\}$$

of matrices. We define an algebra homomorphism

$$\mathcal{A} \to \mathsf{End}(\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})), \quad a \to \hat{a}$$

into the continuous $\mathbb{R}$-linear self maps on the vector space $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ as follows. Let $f \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$, and

$$f(t) = u_1 + iu_2 + \sum_{n=1}^{\infty} w_{1,n}e^{int} + w_{2,n}e^{-int} + iw_{3,n}e^{int} - iw_{4,n}e^{-int}$$

its Fourier series expressed with $w_{i,m} \in \mathbb{R}$. Let us write

$$u = (u_1, u_2)^T \in \mathbb{R}^2, \qquad w_j = (w_{1,j}, w_{2,j}, w_{3,j}, w_{4,j})^T \in \mathbb{R}^4.$$

The action of $\hat{a}$ is defined by

$$(\hat{a}.f)(t) = \tilde{u}_1 + i\tilde{u}_2 + \sum_{n+1}^{\infty} \tilde{w}_{1,n}e^{int} + \tilde{w}_{2,n}e^{-int} + i\tilde{w}_{3,n}e^{int} - i\tilde{w}_{4,n}e^{-int}$$

for $f \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ with $\tilde{u} = B.u$ and $\tilde{w}_j = C.w_j$.

Definition 7.3.2. Define

$$\mathtt{P}_R = \left(\begin{array}{cc|cccc} 1 & & & & & \\ & 0 & & & & \\ \hline & & \frac{1}{2} & \frac{1}{2} & & \\ & & \frac{1}{2} & \frac{1}{2} & & \\ & & & & \frac{1}{2} & \frac{1}{2} \\ & & & & \frac{1}{2} & \frac{1}{2} \end{array}\right), \qquad \mathtt{P}_I = \left(\begin{array}{cc|cccc} 0 & & & & & \\ & 1 & & & & \\ \hline & & \frac{1}{2} & -\frac{1}{2} & & \\ & & -\frac{1}{2} & \frac{1}{2} & & \\ & & & & \frac{1}{2} & -\frac{1}{2} \\ & & & & -\frac{1}{2} & \frac{1}{2} \end{array}\right),$$

$$\mathtt{P}_0 = \left(\begin{array}{cc|cccc} 1 & & & & & \\ & 1 & & & & \\ \hline & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{array}\right), \qquad \mathtt{I} = \left(\begin{array}{cc|cccc} 0 & -1 & & & & \\ 1 & 0 & & & & \\ \hline & & & & -1 & 0 \\ & & & & 0 & 1 \\ & & 1 & 0 & & \\ & & 0 & -1 & & \end{array}\right),$$

$$\mathtt{H} = \left(\begin{array}{cc|cccc} 0 & & & & & \\ & 0 & & & & \\ \hline & & & & -1 & 0 \\ & & & & 0 & -1 \\ & & 1 & 0 & & \\ & & 0 & 1 & & \end{array}\right), \qquad \mathtt{P}_K = \left(\begin{array}{cc|cccc} 0 & & & & & \\ & 0 & & & & \\ \hline & & 0 & 1 & & \\ & & 0 & 1 & & \\ & & & & 0 & 1 \\ & & & & 0 & 1 \end{array}\right),$$

$$
\mathtt{P}_F = \left(\begin{array}{cc|cccc}
0 & & & & & \\
 & 0 & & & & \\
\hline
 & & 1 & & & \\
 & & & 0 & & \\
 & & & & 1 & \\
 & & & & & 0
\end{array}\right).
$$

And define $\hat{\mathtt{P}}_R$, $\hat{\mathtt{P}}_I$, $\hat{\mathtt{I}}$, $\hat{\mathtt{P}}_0$, $\hat{\mathtt{H}}$, $\hat{\mathtt{P}}_F$, and $\hat{\mathtt{P}}_K$ to be the images of the algebra homomorphism defined above (Definition 7.3.1).

Remark: Alternative definitions and charaterization will be provided by

(i) 7.3.8 for $\hat{\mathtt{P}}_F$,
(ii) 7.3.13 for the Hilbert transform $\hat{\mathtt{H}}$,
(iii) 7.3.6 for the projection to the real part $\hat{\mathtt{P}}_R$,
(iv) 7.3.6 for the projection to the imaginary part $\hat{\mathtt{P}}_I$,
(v) 7.3.6 for $\hat{\mathtt{I}}$ to be the multiplication with i,
(vi) 7.3.6 for the projection to the constant Fourier coefficient $\hat{\mathtt{P}}_0$, and
(vii) 7.3.3 for $\hat{\mathtt{P}}_K$.

Lemma 7.3.3. *(i)* $2\hat{\mathtt{P}}_F\hat{\mathtt{P}}_I\hat{\mathtt{P}}_F = \hat{\mathtt{P}}_F$.
(ii) $2\hat{\mathtt{P}}_F\hat{\mathtt{P}}_R\hat{\mathtt{P}}_F = \hat{\mathtt{P}}_F$.
(iii) $\hat{\mathtt{P}}_R = \hat{\mathtt{P}}_0\hat{\mathtt{P}}_R + \hat{\mathtt{P}}_R 2\hat{\mathtt{P}}_F\hat{\mathtt{P}}_R$.
(iv) $\hat{\mathtt{P}}_0\,\hat{\mathtt{H}} = 0$ *and* $(\hat{\mathtt{I}}\hat{\mathtt{H}})^2 = 1 - \hat{\mathtt{P}}_0$.
(v) $\hat{\mathtt{P}}_K(\hat{\mathtt{P}}_R(1-\hat{\mathtt{P}}_0)) = \hat{\mathtt{P}}_R(1-\hat{\mathtt{P}}_0)$.
(vi) $\hat{\mathtt{P}}_R(1-\hat{\mathtt{P}}_0)\hat{\mathtt{P}}_K = \hat{\mathtt{P}}_K$.
(vii) $(1-\hat{\mathtt{P}}_K)(\hat{\mathtt{P}}_0 + \hat{\mathtt{P}}_F) = (\hat{\mathtt{P}}_0 + \hat{\mathtt{P}}_F)$.
(viii) $(\hat{\mathtt{P}}_0 + \hat{\mathtt{P}}_F)(1-\hat{\mathtt{P}}_K) = (1-\hat{\mathtt{P}}_K)$.
(ix) $\hat{\mathtt{P}}_K = 1 - (\hat{\mathtt{P}}_0 + 2\hat{\mathtt{P}}_F\hat{\mathtt{P}}_I)$.
(x) $\hat{\mathtt{P}}_K = (1-\hat{\mathtt{P}}_0)\hat{\mathtt{P}}_R(1-(2\hat{\mathtt{P}}_F + \hat{\mathtt{P}}_0)\hat{\mathtt{P}}_I)$.
(xi) $\hat{\mathtt{P}}_K = 2\hat{\mathtt{P}}_R(1-\hat{\mathtt{P}}_0-\hat{\mathtt{P}}_F)$.
(xii) $\hat{\mathtt{P}}_K\hat{\mathtt{I}} = -\hat{\mathtt{H}}\hat{\mathtt{P}}_K$.
(xiii) $\hat{\mathtt{H}} = \hat{\mathtt{I}}(2\hat{\mathtt{P}}_F - 1 + \hat{\mathtt{P}}_0)$.
(xiv) $2\hat{\mathtt{P}}_R\hat{\mathtt{P}}_F\hat{\mathtt{P}}_R(1-\hat{\mathtt{P}}_0) = \hat{\mathtt{P}}_R(1-\hat{\mathtt{P}}_0)$.

Proof. This lemma can be verified by matrix multiplication of the matrices defined in 7.3.2, and using the algebra homomorphism defined in 7.3.1. □

Lemma 7.3.4. *If $a \in \mathcal{A}$, then*

$$\hat{a} : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad f \mapsto \hat{a}.f$$

is a linear-tame map.

Proof. The operator $\hat{a}$ commutates with the derivative operator on $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$. We can split a into a sum of elementary matrices, which are matrices with one component equal 1 and and the other components equal 0. Let $E \in \mathcal{A}$ be an elementary matrix, then

$$\|\hat{E}.f\|_{L^2} \leq \|f\|_{L^2}$$

for all $f \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \subseteq \mathsf{L}^2(\mathbb{S}^1, \mathbb{C})$, where $\|\cdot\|_{L^2}$ denotes the L^2-norm. Moreover, by Lemma 2.2.4(iv) there exists a real number $C > \mathbb{R}$ such that

$$\|f\|_\infty \leq \|f\|_{L^2} + C\,\|f'\|_{L^2}$$

for all smooth functions $f : \mathbb{S}^1 \to \mathbb{C}$. Here, $\|\cdot\|_\infty$ denotes the sup-norm. We estimate

$$\begin{aligned}
\|\hat{E}.f\|_\infty &\leq \|\hat{E}.f\|_{L^2} + C\,\|(\hat{E}.f)'\|_{L^2} \\
&= \|\hat{E}.f\|_{L^2} + C\,\|\hat{E}.f'\|_{L^2} \\
&\leq \|f\|_{L^2} + C\,\|f'\|_{L^2} \\
&\leq 2\pi\|f\|_\infty + 2\pi C\|f'\|_\infty \\
&\leq 2\pi(1+C)(\|f\|_\infty + \|f'\|_\infty).
\end{aligned}$$

This implies that

$$\begin{aligned}
\|\hat{E}.f\|_n &= \sum_{k=0}^{n} \|\hat{E}.f^{(k)}\|_\infty \\
&\leq 2\pi(1+C)\sum_{k=0}^{n}(\|f^{(k)}\|_\infty + \|f^{(k+1)}\|_\infty) \\
&\leq 4\pi(1+C)\sum_{k=0}^{n+1}\|f^{(k)}\|_\infty \\
&= 4\pi(1+C)\|f\|_{n+1},
\end{aligned}$$

where $\|\cdot\|_n$ denotes the grading defined in 5.1.8. This shows, that $\hat{E}$ is linear-tame according to Definition 5.1.2, and hence $\hat{a}$ is also linear-tame. $\square$

Lemma 7.3.5. *The projections*

$$\hat{\mathtt{P}}_R, \hat{\mathtt{P}}_I, 1-\hat{\mathtt{P}}_0, \hat{\mathtt{P}}_0, \hat{\mathtt{P}}_F, \hat{\mathtt{P}}_K : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$$

are linear-tame, and therefore smooth-tame.

Proof. This is a consequence of Lemma 7.3.4 and Definition 7.3.2. □

Lemma 7.3.6. *(i)* $\hat{\mathtt{P}}_R.\lambda = \mathsf{Re}\circ\lambda.$
(ii) $\hat{\mathtt{P}}_0.\lambda \equiv \frac{1}{2\pi}\int_{\mathbb{S}^1}\lambda(t)dt.$
(iii) $\hat{\mathtt{I}}.\lambda = i\cdot\lambda.$
(iv) $\hat{\mathtt{P}}_I := 1-\hat{\mathtt{P}}_R.$

Proof. (i)-(iii): Let

$$\lambda(t) = u_1 + iu_2 + \sum_{n=1}^{\infty} w_{1,n}e^{int} + w_{2,n}e^{-int} + iw_{3,n}e^{int} - iw_{4,n}e^{-int},$$

then Definition 7.3.1 and Definition 7.3.2 yield

$$\begin{aligned} & (\hat{\mathtt{P}}_R.\lambda)(t) \\ = \;& u_1 + \sum_{n=1}^{\infty}\frac{1}{2}(w_{1,n}+w_{2,n})(e^{int}+e^{-int}) + \frac{i}{2}(w_{3,n}+w_{4,n})(e^{int}-e^{-int}), \end{aligned}$$

$$\begin{aligned} (\hat{\mathtt{P}}_0.\lambda)(t) &= u_1 + iu_2, \quad \text{and} \\ (\hat{\mathtt{I}}.\lambda)(t) &= -u_2 + iu_1 + \sum_{n=1}^{\infty} -w_{3,n}e^{int} + w_{4,n}e^{-int} + iw_{1,n}e^{int} + iw_{2,n}e^{-int}. \end{aligned}$$

(iv): This can be obtained taking into account that $\hat{\mathtt{P}}_I + \hat{\mathtt{P}}_R$ is the unit matrix. □

Lemma 7.3.7. *If*

$$\lambda(t) = \sum_{k\in\mathbb{Z}} a_k e^{ikt},$$

then

$$\begin{aligned} (\hat{\mathtt{P}}_F.\lambda)(t) &= a_1e^{it} + a_2e^{2it} + a_3e^{3it} + \cdots, \\ (\hat{\mathtt{P}}_K.\lambda)(t) &= \cdots + a_{-2}e^{-2it} + a_{-1}e^{-it} + \overline{a_{-1}}e^{it} + \overline{a_{-2}}e^{2it} + \cdots, \textit{and} \\ (\hat{\mathtt{H}}.\lambda)(t) &= \cdots - ia_{-2}e^{-2it} - ia_{-1}e^{-it} + ia_1e^{it} + ia_2e^{2it} + \cdots. \end{aligned}$$

Proof. Let

$$\begin{aligned}\lambda(t) &= \sum_{k\in\mathbb{Z}} a_k e^{ikt} \\ &= u_1 + iu_2 + \sum_{n=1}^{\infty} w_{1,n}e^{int} + w_{2,n}e^{-int} + iw_{3,n}e^{int} - iw_{4,n}e^{-int},\end{aligned}$$

with

$$\begin{aligned} u_1 &= \mathsf{Re}\, a_0, & w_{1,n} &:= \mathsf{Re}\, a_n, & w_{2,n} &= \mathsf{Re}\, a_{-n}, \\ u_2 &= \mathsf{Im}\, a_0, & w_{3,n} &:= \mathsf{Im}\, a_n, & w_{4,n} &= -\mathsf{Im}\, a_{-n}. \end{aligned}$$

Applying Definition 7.3.1 and Definition 7.3.2 provides us with

$$(\hat{\mathtt{P}}_F.\lambda)(t) = \sum_{n=1}^{\infty} w_{1,n}e^{int} + iw_{3,n}e^{int} = \sum_{n=1}^{\infty} a_n e^{int},$$

$$\begin{aligned}(\hat{\mathtt{P}}_K.\lambda)(t) &= \sum_{n=1}^{\infty} w_{2,n}e^{int} + w_{2,n}e^{-int} + iw_{4,n}e^{int} - iw_{4,n}e^{-int} \\ &= \sum_{n=1}^{\infty} a_{-n}e^{-int} + \overline{a_{-n}}e^{int},\end{aligned}$$

and

$$\begin{aligned}(\hat{\mathtt{H}}.\lambda)(t) &= \sum_{n=1}^{\infty} -w_{3,n}e^{int} - w_{4,n}e^{-int} + iw_{1,n}e^{int} - iw_{2,n}e^{-int} \\ &= \sum_{n=1}^{\infty} ia_n e^{int} - ia_{-n}e^{-int}.\end{aligned}$$

□

Lemma 7.3.8. *We have*

$$(\hat{\mathtt{P}}_F.\lambda)(\tau) = \frac{1}{2\pi}\int_{\mathbb{S}^1} \frac{\lambda(\tau)-\lambda(\theta)}{1-e^{-i(\tau-\theta)}} d\theta$$

for $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C})$ *and* $\in \mathbb{S}^1$.

Proof. Let

$$\lambda(t) = u_1 + iu_2 + \sum_{n=1}^{\infty} w_{1,n}e^{int} + w_{2,n}e^{-int} + iw_{3,n}e^{int} - iw_{4,n}e^{-int},$$

then the definitions 7.3.1 and 7.3.2 imply

$$(\hat{\mathtt{P}}_F.\lambda)(t) = \sum_{n=1}^{\infty} w_{1,n}e^{int} + iw_{3,n}e^{int}.$$

Let $\lambda_n(t) = e^{int}$, then we have

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{S}^1} \frac{\lambda_n(\tau)-\lambda_n(\theta)}{1-e^{-i(\tau-\theta)}}d\theta &= \frac{1}{2\pi}e^{in\tau}\int_{\mathbb{S}^1}\frac{1-e^{-in(\tau-\theta)}}{1-e^{-i(\tau-\theta)}}d\theta \\
&= \frac{1}{2\pi}e^{in\tau}\sum_{k=0}^{n-1}\int_{\mathbb{S}^1} e^{-ik(\tau-\theta)}d\theta \\
&= e^{in\tau}
\end{aligned}$$

for $n \geq 1$, and

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{S}^1} \frac{\lambda_n(\tau)-\lambda_n(\theta)}{1-e^{-i(\tau-\theta)}}d\theta &= \frac{1}{2\pi}e^{in\tau}\int_{\mathbb{S}^1} e^{i(\tau-\theta)}\frac{1-e^{-in(\tau-\theta)}}{e^{i(\tau-\theta)}-1}d\theta \\
&= -\frac{1}{2\pi}e^{in\tau}\sum_{k=1}^{-n}\int_{\mathbb{S}^1} e^{ik(\tau-\theta)}d\theta \\
&= 0
\end{aligned}$$

for $n \leq -1$. If $n = 0$, then the integrand vanishes, because λ is constant. This shows that the lemma is fulfilled if λ has only a finite number of non-zero Fourier coefficients. By Proposition 5.4.4, the mapping

$$\mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}), \quad \lambda \mapsto \{\tau \mapsto \frac{i}{2\pi}\int_{\mathbb{S}^1}\frac{\lambda(\theta)-\lambda(\tau)}{f(\theta)-f(\tau)}f'(\theta)d\theta\}$$

is continuous. The substitution $f(t) = e^{-it}$ yields

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{\lambda(\tau)-\lambda(\theta)}{f(\theta)-f(\tau)}f'(\theta)\,d\theta &= \frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{\lambda(\tau)-\lambda(\theta)}{e^{-i\theta}-e^{-i\tau}}(-i)e^{-i\theta}\,d\theta \\
&= -\frac{i}{2\pi}\int_{\mathbb{S}^1}\frac{\lambda(\tau)-\lambda(\theta)}{1-e^{-i(\tau-\theta)}}\,d\theta,
\end{aligned}$$

and we obtain that

$$\mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}), \quad \lambda \mapsto \{\tau \mapsto \frac{i}{2\pi}\int_{\mathbb{S}^1}\frac{\lambda(\theta)-\lambda(\tau)}{1-e^{-i(\tau-\theta)}}d\theta\}$$

is continuous. Hence, the lemma is proved for all $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C})$. $\square$

Lemma 7.3.9. *(i) The projection $\hat{\mathtt{P}}_F : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is a projection onto the subspace of functions $\lambda : \mathbb{S}^1 \to \mathbb{C}$ with the form*

$$\lambda(t) = a_1 e^{it} + a_2 e^{2it} + \cdots .$$

(ii) The projection $1 - \hat{\mathtt{P}}_F : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is a projection onto the subspace of functions $\lambda : \mathbb{S}^1 \to \mathbb{C}$ with the form

$$\lambda(t) = b_0 + b_1 e^{-it} + b_2 e^{-2it} + \cdots .$$

Proof. It is a consequence of Lemma 7.3.7. □

Lemma 7.3.10. $(\hat{\mathtt{P}}_0 + \hat{\mathtt{P}}_F).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) = \mathtt{F}^+$.

Proof. By Definition 2.2.3, the subset $\mathtt{F}^+ \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is the subset of functions of the form

$$f(t) = a_0 + a_1 e^{it} + a_2 e^{2it} + \cdots$$

with $a_0, a_1, a_2, \ldots \in \mathbb{C}$. By Lemma 7.3.9, the projection $\hat{\mathtt{P}}_F$ projects onto the subset of function with coefficients whose index is > 0. And by Lemma 7.3.6 the projections $\hat{\mathtt{P}}_0$ projects out the constant coefficient a_0. □

Definition 7.3.11. We define $\mathtt{E}_0 \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ to be the subset of the functions with Fourier series of the form

$$f(t) = r + a_1 e^{i\tau} + a_2 e^{2i\tau} + \cdots$$

with $r \in \mathbb{R}$ and $a_1, a_2, \ldots \in \mathbb{C}$.

Lemma 7.3.12. $\mathtt{E}_0 = (\hat{\mathtt{P}}_R \hat{\mathtt{P}}_0 + \hat{\mathtt{P}}_F).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$.

Proof. According to Definition 7.3.2, we have

$$\mathtt{P}_R \mathtt{P}_0 + \mathtt{P}_F = \left(\begin{array}{cc|cccc} 1 & & & & & \\ & 0 & & & & \\ \hline & & 1 & 0 & & \\ & & 0 & 0 & & \\ & & & & 1 & 0 \\ & & & & 0 & 0 \end{array} \right)$$

and the statement is a consequence of the Definition 7.3.1. □

Lemma 7.3.13 (Hilbert transform)**.** *We have*

$$(\hat{\mathtt{H}}.\lambda)(\tau) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \frac{\lambda(\tau) - \lambda(\theta)}{\tan \frac{\tau - \theta}{2}} d\theta$$

for all $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ and $t \in \mathbb{S}^1$.

Remark: The mapping

$$\hat{\mathtt{H}} : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \lambda \mapsto \hat{\mathtt{H}}.\lambda$$

is called *Hilbert transform* [16].

Proof. By Lemma 7.3.3(xiii), we have

$$\hat{\mathtt{H}} = \hat{\mathtt{I}}(2\hat{\mathtt{P}}_F - 1 + \hat{\mathtt{P}}_0),$$

and we can compute

$$\begin{aligned}
(\hat{\mathtt{H}}.\lambda)(\tau) &= i\left\{2\,\frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{\lambda(\tau)-\lambda(\theta)}{1-e^{-i(\tau-\theta)}}d\theta - \lambda(\tau) + \frac{1}{2\pi}\int_{\mathbb{S}^1}\lambda(\theta)\,d\theta\right\}\\
&= \frac{i}{2\pi}\int_{\mathbb{S}^1}(\lambda(\tau)-\lambda(\theta))\left\{\frac{2}{1-e^{-i(\tau-\theta)}}-1\right\}d\theta\\
&= \frac{1}{2\pi}\int_{\mathbb{S}^1}(\lambda(\tau)-\lambda(\theta))\,i\,\frac{1+e^{-i(\tau-\theta)}}{1-e^{-i(\tau-\theta)}}\,d\theta\\
&= \frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{\lambda(\tau)-\lambda(\theta)}{\tan\frac{\tau-\theta}{2}}d\theta.
\end{aligned}$$

□

Lemma 7.3.14. *If $P, Q : V \to V$ are two projections with $PQ = Q$ and $QP = P$, then*

$$P.V = Q.V.$$

Proof. We have $Q.V = P.Q.V \subseteq P.V$ and vice verse $P.V \subseteq Q.V$. □

Proposition 7.3.15. *We have*

(i) $\hat{\mathtt{P}}_K.\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) = \hat{\mathtt{P}}_R(1-\hat{\mathtt{P}}_0).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$, *and*

(ii) $(1-\hat{\mathtt{P}}_K).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) = (\hat{\mathtt{P}}_0 + \hat{\mathtt{P}}_F).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$.

Proof. (i): By Lemma 7.3.3(v)+(vi), we have

$$\begin{aligned}
\hat{\mathtt{P}}_K(\hat{\mathtt{P}}_R(1-\hat{\mathtt{P}}_0)) &= \hat{\mathtt{P}}_R(1-\hat{\mathtt{P}}_0) \qquad \text{and}\\
\hat{\mathtt{P}}_R(1-\hat{\mathtt{P}}_0)\hat{\mathtt{P}}_K &= \hat{\mathtt{P}}_K.
\end{aligned}$$

The Substitution $P = \hat{\mathtt{P}}_K$ and $Q = \hat{\mathtt{P}}_R(1-\hat{\mathtt{P}}_0)$ give us

$$P.Q = Q \qquad \text{and} \qquad Q.P = P,$$

such that we can apply Lemma 7.3.14 and get

$$\hat{\mathtt{P}}_K.\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) = \hat{\mathtt{P}}_R(1-\hat{\mathtt{P}}_0).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}).$$

(ii): By Lemma 7.3.3(vii)+(viii), we have

$$\begin{aligned}(1-\hat{\mathtt{P}}_K)(\hat{\mathtt{P}}_0+\hat{\mathtt{P}}_F) &= (\hat{\mathtt{P}}_0+\hat{\mathtt{P}}_F), \text{ and}\\ (\hat{\mathtt{P}}_0+\hat{\mathtt{P}}_F)(1-\hat{\mathtt{P}}_K) &= (1-\hat{\mathtt{P}}_K).\end{aligned}$$

Hence, the same procedure as above yields

$$(1-\hat{\mathtt{P}}_K).\mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) = (\hat{\mathtt{P}}_0+\hat{\mathtt{P}}_F).\mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}).$$

□

7.4 Smaller Neighborhood $\mathtt{O}_f$

The clue of this chapter is to show that the function

$$\hat{\theta}_f : \mathtt{U}_f \to \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{R})$$

is a tame diffeomorphic embedding. We show this with the Nash-Moser Theorem. To verify the assumptions of the Nash-Moser Theorem, we have to invert the derivative

$$d\hat{\theta}_f : (\mathtt{U}_f \cap \mathtt{V}^E) \times \mathtt{E}_1 \to \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{R})$$

with respect to the right argument, and show that this inverse is smooth-tame. Actually, we do not engage the whole neighborhood $\mathtt{U}_f \cap \mathtt{V}^E$, but restrict to a smaller one, which will be denoted by $\mathtt{O}_f^E$ (Definition 7.4.4). It turns out that this restriction to $\mathtt{O}_f^E$ is necessary for the proof of Lemma 7.5.6 to go through. In this very proof, the reciprocal of $\mathtt{Re}(\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(\eta))))$ occurs in the inverse of a particular function. In this section, we will show that $\mathtt{O}_f^E$ is an open neighborhood of $\hat{\Lambda}(f)$. Lemma 7.4.12 guarantees that $\mathtt{O}_f^E$ is an open subset of $\mathtt{V}^E$ and Lemma 7.4.15 asserts that $\hat{\Lambda}(f) \in \mathtt{O}_f^E$.

Definition 7.4.1. We define

$$\hat{l}(f)(t) := \int_0^t \frac{f'(\theta)}{f(\theta)}d\theta - \frac{1}{2\pi}\int_{\mathbb{S}^1}\int_0^\tau \frac{f'(\theta)}{f(\theta)}d\theta d\tau$$

for $t \in \mathbb{S}^1$ and $f \in \mathtt{W}$. Where the open subset $\mathtt{W} \subseteq \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C})$ was defined in 7.2.3.

Remark: The function $\hat{l}(f)$ is 2π-periodic, because the winding number of f is

$$\mathtt{w}(f) = \int_{\mathbb{S}^1} \frac{f'(\theta)}{f(\theta)}d\theta = 0.$$

Definition 7.4.2. Let $f \in \mathtt{W}$. We define $\hat{\mathtt{m}}(f)(\tau) := f(\tau) \cdot e^{-\hat{l}(f)(\tau)}$.

Lemma 7.4.3. *For $f \in \mathtt{V}$ and $\eta \in \mathtt{U}_f$ the expression $\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(\eta)))$ makes sense.*

Proof. By Lemma 7.1.5, we have $\hat{\Lambda}(\eta) \in \mathtt{U}_f$. Therefore, we have $\hat{k}_f(\hat{\Lambda}(\eta)) \in \mathtt{W}$ due to Lemma 7.2.9. Hence, the expression $\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(\eta)))$ makes sense by Definition 7.4.2. □

This lemma admits the following definition.

Definition 7.4.4. For $f \in \mathtt{V}$ we define the subsets

(i) $\mathtt{O}_f := \{\, \eta \in \mathtt{V} \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \;:\; \mathsf{Re}(\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(\eta)))) \neq 0 \wedge \eta \in \mathtt{U}_f \,\}$ and

(ii) $\mathtt{O}_f^E := \{\, \eta \in \mathtt{V}^E \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \;:\; \mathsf{Re}(\hat{\mathtt{m}}(\hat{k}_f(\eta))) \neq 0 \wedge \eta \in \mathtt{U}_f \,\}$.

Remark: Since $\hat{k}_f(\eta) \in \mathtt{W}$ holds by Lemma 7.2.9, the term $\hat{\mathtt{m}}(\hat{k}_f(\eta))$ is defined correctly.

Lemma 7.4.5. *If $f \in \mathtt{W}$, then*

(i) $\hat{l}(f) = (1 - \hat{\mathtt{P}}_0).\{t \mapsto \int_0^t \frac{f'(\theta)}{f(\theta)} d\theta\}$, *and*

(ii) $\hat{\mathtt{P}}_0.\hat{l}(f) = 0$.

Proof. This is a consequence of Lemma 7.3.6(ii), which asserts

$$(\hat{\mathtt{P}}_0.\lambda) \equiv \frac{1}{2\pi} \int_{\mathbb{S}^1} \lambda(t) dt$$

for all $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$. □

Lemma 7.4.6. *The map $\hat{l} : \mathtt{W} \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is smooth-tame.*

Proof. By Lemma 5.2.10(i), the derivative map

$$\mathtt{V} \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad f \mapsto f'$$

is smooth-tame. Since by Lemma 5.3.6(iii) the quotient map is smooth tame, we have that

$$\mathtt{V} \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad f \mapsto \frac{f'}{f}$$

is smooth-tame. Taking into account that by Lemma 5.2.10(ii) the integration map is smooth tame, we conclude that

$$\hat{l} : \mathtt{W} \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad f \mapsto (1 - \hat{\mathtt{P}}_0).\{t \mapsto \int_0^t \frac{f'(\theta)}{f(\theta)} d\theta\}$$

is smooth-tame. □

Lemma 7.4.7. *If $f, g \in \mathtt{W}$, then*
(i) $\hat{l}(f \cdot g) = \hat{l}(f) + \hat{l}(g)$, *and*
(ii) $\hat{l}(1/f) = -\hat{l}(f)$.

Proof. This follows from Definition 7.4.1. □

Lemma 7.4.8. *If $f \in \mathtt{W} \cap \mathtt{F}^+$, then $\hat{l}(f) = \hat{\mathtt{P}}_F.\hat{l}(f)$.*

Proof. Assume $f \in \mathtt{F}^+$, then $f(t) = a_0 + a_1 e^{it} + a_2 e^{2it} + \cdots$ by Definition 2.2.3, and therefore

$$\frac{f'(t)}{f(t)} = \frac{ia_1 e^{it} + \cdots}{a_0 + a_1 e^{it} + \cdots} = c_0 + c_1 e^{it} + c_2 e^{2it} + \cdots .$$

Let us consider

$$\hat{l}(f)(t) := \int_0^t \frac{f'(\theta)}{f(\theta)} d\theta - \frac{1}{2\pi} \int_{\mathbb{S}^1} \int_0^\tau \frac{f'(\theta)}{f(\theta)} d\theta d\tau.$$

The integration

$$\int_0^t \frac{f'(t)}{f(t)} dt$$

does not change the form of the Fourier series and the second summand

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} \int_0^\tau \frac{f'(\theta)}{f(\theta)} d\theta d\tau$$

is just the constant Fourier coefficient c_0. Hence, the form of $\hat{l}(f)(t)$ is

$$\hat{l}(f)(t) = d_1 e^{it} + d_2 e^{2it} + \cdots .$$

By Lemma 7.3.9, the projection $\hat{\mathtt{P}}_F$ is exactly the projection onto the subspace of functions of this form. Hence, the statement is proved. □

Lemma 7.4.9. *If $f \in \mathtt{W}$, then*

(i) $\hat{\mathtt{m}}(f) : \mathbb{S}^1 \to \mathbb{C}$, $\tau \mapsto \hat{\mathtt{m}}(f)(\tau)$ *is the constant map, and*

(ii) $\hat{\mathtt{m}} : \mathtt{W} \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$, $f \mapsto \hat{\mathtt{m}}(f)$ *is smooth-tame.*

Proof. (i):

$$\begin{aligned}
\hat{\mathtt{m}}(f)'(t) &\overset{7.4.2}{=} \frac{d}{dt}[f(t) \cdot e^{-\hat{l}(f)(t)}] = \; f'(t) \cdot e^{-\hat{l}(f)(t)} + f(t) \cdot [-\frac{d}{dt}\hat{l}(f)(t)]e^{-\hat{l}(f)(t)} \\
&\overset{7.4.1}{=} f'(t) \cdot e^{-\hat{l}(f)(t)} + f(t) \cdot [-\frac{f'(t)}{f(t)}]e^{-\hat{l}(f)(\tau)} \\
&= 0.
\end{aligned}$$

(ii): The function

$$\hat{l} : \mathtt{W} \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad f \mapsto \hat{l}(f)$$

is smooth-tame by Lemma 7.4.6, and the function

$$\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \lambda \mapsto e^\lambda$$

is smooth-tame by Lemma 5.3.5. Furthermore, the multiplication map

$$\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad (f, g) \mapsto f \cdot g$$

is smooth-tame by Lemma 5.3.6(i). We conclude that

$$\hat{\mathtt{m}} : \mathtt{W} \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad f \mapsto f \cdot e^{-\hat{l}(f)}$$

is smooth-tame, because the composition of the smooth-tame maps are smooth-tame by Lemma 5.2.8. □

Lemma 7.4.10. *If $f, g \in \mathtt{W}$, then*

(i) $\hat{\mathtt{m}}(f \cdot g) = \hat{\mathtt{m}}(f) \cdot \hat{\mathtt{m}}(g)$,

(ii) $\hat{\mathtt{m}}(1/f) = 1/\hat{\mathtt{m}}(f)$, *and*

(iii) $f \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$ *implies* $\hat{\mathtt{m}}(f) \in \mathbb{R}$.

Proof. (i)+(ii): These are just consequences of Lemma 7.4.7 and Definition 7.4.2(i).
(iii): It follows from Definition 7.4.1 and Definition 7.4.2(i). □

Lemma 7.4.11. *If* $f \in \mathtt{W} \cap \mathtt{F}^+$*, then* $\hat{\mathtt{m}}(f) = a_0$*. Here* a_0 *denotes the constant Fourier coefficient of the function* $f : \mathbb{S}^1 \to \mathbb{C}$*.*

Proof. From Lemma 7.4.8 we deduce $\hat{l}(f) \in \hat{\mathtt{P}}_F.\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ and the Fourier series is of the form

$$\hat{l}(f)(t) = b_1 e^{it} + b_2 e^{2it} + \cdots .$$

By Lemma 2.3.9, the Fourier series of $\mathsf{exp}(\hat{l}(f)(t))$ is

$$e^{\hat{l}(f)(t)} = 1 + c_1 e^{it} + c_2 e^{2it} + \cdots .$$

Definition 7.4.2(i) yields

$$f(t) = \hat{\mathtt{m}}(f) \cdot e^{\hat{l}(f)(t)} \hat{\mathtt{m}}(f) \cdot (1 + c_1 e^{it} + c_2 e^{2it} + \cdots),$$

and we get the result comparing the Fourier coefficients. $\square$

Remark:

$$\hat{\mathtt{m}}(f) = f(0) \cdot e^{-\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\tau} \frac{f'(\theta)}{f(\theta)} d\theta d\tau}.$$

Lemma 7.4.12. *If* $f \in \mathtt{V}$*, then* $\mathtt{O}_f^E$ *is an open subset of* $\mathtt{V}^E$*. Furthermore,* $\mathtt{O}_f^E$ *is an open subset of* $\mathtt{E}_1$*.*

Proof. The function

$$\alpha : \mathtt{U}_f \to \mathbb{R}, \quad \eta \mapsto \mathsf{Re}(\hat{\mathtt{m}}(\hat{k}_f(\eta)))$$

is continuous because $\hat{k}_f : \mathtt{U}_f \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is continuous by Lemma 7.2.10, and $\hat{\mathtt{m}} : \mathtt{V} \to \mathbb{C}$ is continuous by Lemma 7.4.9(ii). The preimage $\alpha^{-1}(\mathbb{R}\backslash\{0\})$ is an open subset of $\mathtt{U}_f$, and therefore an open subset of $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$, because by Lemma 7.1.5 the subset $\mathtt{U}_f \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is open. Hence,

$$\mathtt{O}_f^E = \mathtt{V}^E \cap \alpha^{-1}(\mathbb{R}\backslash\{0\})$$

is an open subset of $\mathtt{V}^E$. Furthermore, since $\mathtt{V}^E \subseteq \mathtt{E}_1$ is open by Lemma 7.2.2, the subset $\mathtt{O}_f^E \subsetneq \mathtt{E}_1$ is also open. $\square$

The following three lemmas show that $\mathtt{O}_f^E$ is a neighborhood of $\hat{\Lambda}(f)$.

Lemma 7.4.13. *If $f \in \mathtt{V}$, then*

$$\hat{\mathtt{m}}(\{t \mapsto -i\hat{\Lambda}(f)' \cdot e^{-it}\}) \in \mathbb{R}.$$

Proof. The Fourier series of $\hat{\Lambda}(f) \in \mathtt{V}^E$ is

$$\hat{\Lambda}(f)(t) = e^{it}(r + a_1 e^{it} + a_2 e^{2it} + \cdots)$$

with $r \in \mathbb{R}$ and $a_1, a_2, \ldots \in \mathbb{C}$. This implies

$$-i\hat{\Lambda}(f)'(t) \cdot e^{-it} = r + 2a_1 e^{it} + 3a_2 e^{2it} + \cdots$$

and with Lemma 7.4.11 we conclude

$$\hat{\mathtt{m}}(\{t \mapsto -i\hat{\Lambda}(f)' \cdot e^{-it}\}) = r \in \mathbb{R}.$$

□

Lemma 7.4.14. *If $f \in \mathtt{V}$, then $\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f))) \in \mathbb{R}\backslash\{0\}$.*

Proof. We compute

$$\begin{aligned} & \hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f))) \\ \overset{7.2.8(ii)}{=} \; & \hat{\mathtt{m}}(\{t \mapsto i\frac{|\hat{\Lambda}(f)'(t) \cdot e^{-it}|}{\hat{\Lambda}(f)'(t) \cdot e^{-it}}\}) \\ \overset{7.4.10(i)+(ii)}{=} \; & \hat{\mathtt{m}}(|\hat{\Lambda}(f)' \cdot e^{-it}|) \cdot \hat{\mathtt{m}}(\{t \mapsto -i\hat{\Lambda}(f)'(t) \cdot e^{-it}\})^{-1}. \end{aligned}$$

Moreover, we have

$$\hat{\mathtt{m}}(|\hat{\Lambda}(f)' \cdot e^{-it}|) \in \mathbb{R}$$

by Lemma 7.4.10(iii), and

$$\hat{\mathtt{m}}(\{t \mapsto -i\hat{\Lambda}(f)'(t) \cdot e^{-it}\}) \in \mathbb{R}$$

by Lemma 7.4.13. Combining these three statements results in

$$\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f))) \in \mathbb{R}.$$

By Lemma 7.2.9 and Definition 7.4.2, we obtain $\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f))) \neq 0$. Hence, the lemma is proved. □

Lemma 7.4.15. *If $f \in \mathtt{V}$, then $\hat{\Lambda}(f) \in \mathtt{O}_f^E$.*

Proof. By Definition 6.1.3, we have $f(\mathbb{S}^1) \subseteq \mathtt{A}_f$ and therefore

$$\hat{\Lambda}(f)(t) \overset{4.2.3(i)}{=} f(\hat{\Gamma}(f)^{-1}(t)) \in \mathtt{A}_f$$

for all $t \in \mathbb{S}^1$. The derivative of

$$\pi_2 \circ \mathtt{N}_f^{-1} \circ \hat{\Lambda}(f) = \hat{\Sigma}_f(\hat{\Lambda}(f)) = \hat{\Gamma}(f)^{-1}$$

is never zero, because $\hat{\Gamma}(f) \in \mathsf{Diff}^+\mathbb{S}^1$. Hence, by Definition 7.1.1 we get

$$\hat{\Lambda}(f) \in \mathtt{U}_f.$$

By Lemma 7.4.14, we have

$$\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f))) \in \mathbb{R}\backslash\{0\},$$

which implies

$$\mathsf{Re}(\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f)))) \neq 0.$$

Using the Definition 7.4.4(ii), we obtain $\hat{\Lambda}(f) \in \mathtt{O}_f^E$. □

7.5 Inverse of $d\hat{\theta}_f$

In this section, we will show that the inverse of the derivative of $\hat{\theta}_f$ with respect to the right argument is smooth-tame (Proposition 7.5.8). The strategy is that we write the derivative of $\hat{\theta}_f$ as a composition of the three maps $\hat{\delta}_f$, $\hat{\vartheta}_f$ and $\hat{\sigma}_f$. We compute the inverse with respect to the right variable of these three functions explicitly. With these formulas at hand it is easy to verify that $\hat{\delta}_f^{\#}$, $\hat{\vartheta}_f^{\#}$, and $\hat{\sigma}_f^{\#}$ are smooth-tame.

Definition 7.5.1. Let $f \in \mathtt{V}$. We define

(i)

$$\hat{\delta}_f(\eta,\lambda)(\tau) := e^{2[\hat{\mathtt{P}}_F.\hat{\mathtt{P}}_I.\hat{l}(\hat{k}_f(\eta))](\tau)} \cdot e^{-i\tau}\lambda(\tau),$$

(ii)

$$\hat{\vartheta}_f(\eta,\lambda)(\tau) := \mathsf{Re}(\hat{\mathtt{m}}(\hat{k}_f(\eta)) \cdot \lambda(\tau)), \quad \text{and}$$

(iii)

$$\hat{\sigma}_f(\eta,\lambda)(t) := e^{[\hat{\mathtt{P}}_K\hat{l}(\hat{k}_f(\eta))](\tau)} \cdot \lambda(\tau)\big|_{\tau=[\hat{\Sigma}_f(\eta)]^{-1}(t)}$$

for $\eta \in \mathtt{U}_f$ and $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C})$.

In the succeding, we show that

$$\begin{aligned}
\hat{\delta}_f &: \mathtt{O}_f^E \times \mathtt{E}_1 \to \mathtt{E}_0, \\
\hat{\vartheta}_f &: \mathtt{O}_f^E \times \mathtt{E}_0 \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}), \quad \text{and} \\
\hat{\sigma}_f &: \mathtt{O}_f^E \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})
\end{aligned}$$

define functions, invert them with respect to the right argument, and show that they are smooth-tame.

Lemma 7.5.2. *If $f \in \mathtt{V}$, then $d\hat{\theta}_f(\eta, \lambda) = \hat{\sigma}_f(\eta, \hat{\vartheta}_f(\eta, \hat{\delta}_f(\eta, \lambda)))$ for $\eta \in \mathtt{U}_f$ and $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$.*

Proof. We compute

$$\begin{aligned}
& d\hat{\theta}_f(\eta, \lambda)(\hat{\Sigma}_f(\eta)(\tau)) \\
\overset{7.2.7}{=}\ & \mathsf{Re}(\hat{k}_f(\eta)(\tau) \cdot e^{-i\tau}\lambda(\tau)) \\
\overset{7.4.5}{=}\ & \mathsf{Re}(e^{[(1-\hat{\mathtt{P}}_0).\hat{l}(\hat{k}_f(\eta))](\tau)} \cdot \hat{k}_f(\eta)(\tau) e^{-\hat{l}(\hat{k}_f(\eta))(\tau)} \cdot e^{-i\tau}\lambda(\tau)) \\
\overset{7.4.2}{=}\ & \mathsf{Re}(e^{[(1-\hat{\mathtt{P}}_0).\hat{l}(\hat{k}_f(\eta))](\tau)} \cdot \hat{\mathtt{m}}(\hat{k}_f(\eta)) \cdot e^{-i\tau}\lambda(\tau)) \\
\overset{7.3.3(ix)}{=}\ & \mathsf{Re}(e^{[(\hat{\mathtt{P}}_K + 2\hat{\mathtt{P}}_F.\hat{\mathtt{P}}_I).\hat{l}(\hat{k}_f(\eta))](\tau)} \cdot \hat{\mathtt{m}}(\hat{k}_f(\eta)) \cdot e^{-i\tau}\lambda(\tau)) \\
=\ & \mathsf{Re}(e^{[\hat{\mathtt{P}}_K \hat{l}(\hat{k}_f(\eta))](\tau)} \cdot \hat{\mathtt{m}}(\hat{k}_f(\eta)) \cdot e^{2[\hat{\mathtt{P}}_F.\hat{\mathtt{P}}_I.\hat{l}(\hat{k}_f(\eta))](\tau)} \cdot e^{-i\tau}\lambda(\tau)) \\
\overset{7.3.15(i)}{=}\ & e^{[\hat{\mathtt{P}}_K \hat{l}(\hat{k}_f(\eta))](\tau)} \cdot \mathsf{Re}(\hat{\mathtt{m}}(\hat{k}_f(\eta)) \cdot e^{2[\hat{\mathtt{P}}_F.\hat{\mathtt{P}}_I.\hat{l}(\hat{k}_f(\eta))](\tau)} \cdot e^{-i\tau}\lambda(\tau)) \\
\overset{7.5.1(i)}{=}\ & e^{[\hat{\mathtt{P}}_K \hat{l}(\hat{k}_f(\eta))](\tau)} \cdot \mathsf{Re}(\hat{\mathtt{m}}(\hat{k}_f(\eta)) \cdot \hat{\delta}_f(\eta, \lambda)(\tau)) \\
\overset{7.5.1(ii)}{=}\ & e^{[\hat{\mathtt{P}}_K \hat{l}(\hat{k}_f(\eta))](\tau)} \cdot \hat{\vartheta}_f(\eta, \hat{\delta}_f(\eta, \lambda))(\tau) \\
\overset{7.5.1(iii)}{=}\ & \hat{\sigma}_f(\eta, \hat{\vartheta}_f(\eta, \hat{\delta}_f(\eta, \lambda)))(\hat{\Sigma}_f(\eta)(\tau)).
\end{aligned}$$

Taking into account that $\hat{\Sigma}_f(\eta) \in \mathsf{Diff}^+\mathbb{S}^1$ is a diffeomorphism, we get the result. □

Lemma 7.5.3. *The functions*

(i) $\mathtt{O}_f^E \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \eta \mapsto e^{-2\hat{\mathtt{P}}_F.\hat{\mathtt{P}}_I.\hat{l}(\hat{k}_f(\eta))}$, *and*

(ii) $\mathtt{O}_f^E \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \eta \mapsto e^{-\hat{\mathtt{P}}_K.\hat{l}(\hat{k}_f(\eta))}$

are smooth-tame.

Proof. By Lemma 7.4.6 the map

$$\hat{l} : \mathtt{W} \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$$

is smooth-tame and by Lemma 7.2.10 the map

$$\hat{k}_f : \mathtt{U}_f \to \mathtt{W} \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$$

is smooth-tame for $f \in \mathtt{V}$. Due to Lemma 7.3.5, the three projections $\hat{\mathtt{P}}_F$, $\hat{\mathtt{P}}_I$ and $\hat{\mathtt{P}}_K$ are linear-tame. From the statements above, we obtain that the functions

$$\begin{aligned} &\mathtt{O}_f^E \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \qquad \eta \mapsto -2\hat{\mathtt{P}}_F.\hat{\mathtt{P}}_I.\hat{l}(\hat{k}_f(\eta)), \qquad \text{and} \\ &\mathtt{O}_f^E \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \qquad \eta \mapsto -\hat{\mathtt{P}}_K.\hat{l}(\hat{k}_f(\eta)) \end{aligned}$$

are smooth-tame. Since the exponential function $\mathbb{C} \to \mathbb{C}$, $t \mapsto e^t$ is smooth, the left composition with the exponential function is smooth-tame by Lemma 5.3.5. Hence, we get the results. □

Recall, that $\mathtt{E}_0 \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ was defined in 7.3.11 to be the subset of the functions of the form

$$f(t) = r + a_1 e^{i\tau} + a_2 e^{2i\tau} + \cdots,$$

and $\mathtt{E}_1 \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ was defined in 7.2.1 to be the subset of functions of the form

$$f(t) = e^{it}(r + a_1 e^{it} + a_2 e^{2it} + \cdots)$$

with $r \in \mathbb{R}$ and $a_1, a_2, \ldots \in \mathbb{C}$.

Lemma 7.5.4. *Let $f \in \mathtt{V}$. Then the inverse mapping of*

$$\hat{\delta}_f : \mathtt{O}_f^E \times \mathtt{E}_1 \to \mathtt{E}_0, \quad (\eta, \lambda) \mapsto \hat{\delta}_f(\eta, \lambda),$$

$$\hat{\delta}_f(\eta, \lambda)(\tau) := e^{2[\hat{\mathtt{P}}_F.\hat{\mathtt{P}}_I.\hat{l}(\hat{k}_f(\eta))](\tau)} \cdot e^{-i\tau}\lambda(\tau)$$

with respect to the right argument is

$$\hat{\delta}_f^{\#} : \mathtt{O}_f^E \times \mathtt{E}_0 \to \mathtt{E}_1,$$

$$\hat{\delta}_f^{\#}(\eta, \lambda)(\tau) = e^{-2[\hat{\mathtt{P}}_F.\hat{\mathtt{P}}_I.\hat{l}(\hat{k}_f(\eta))](\tau)} \cdot e^{i\tau}\lambda(\tau)$$

and is smooth-tame.

Proof. 1. Step: Correctness of the domains:
By Lemma 7.3.9(i), the Fourier series of the exponent is

$$-2[\hat{\mathtt{P}}_F.\hat{\mathtt{P}}_I.\hat{l}(\hat{k}_f(\eta))](\tau) = b_1 e^{it} + b_2 e^{2it} + \cdots .$$

and Lemma 2.3.9 yields

$$e^{-2[\hat{\mathtt{P}}_F.\hat{\mathtt{P}}_I.\hat{l}(\hat{k}_f(\eta))](\tau)} = 1 + a_1 e^{it} + a_2 e^{2it} + \cdots .$$

This implies that

$$\mathtt{E}_0 \to \mathtt{E}_0, \quad \lambda \mapsto e^{-2[\hat{\mathtt{P}}_F.\hat{\mathtt{P}}_I.\hat{l}(\hat{k}_f(\eta))]} \cdot \lambda$$

is a bijective map. By the definition of $\mathtt{E}_0$ and $\mathtt{E}_1$, we obtain that

$$\mathtt{E}_1 \to \mathtt{E}_0, \quad \lambda \mapsto \{\tau \mapsto e^{-i\tau}\lambda(\tau)\}$$

is bijective. Combining this, we have checked the domains.
2. Step: Tameness:
By Lemma 5.3.6(i), the map

$$\mathtt{E}_0 \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \lambda \mapsto \{\tau \mapsto e^{i\tau}\lambda(\tau)\}$$

is smooth-tame and by Lemma 7.5.3 the map

$$\mathtt{O}_f^E \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \eta \mapsto e^{-2[\hat{\mathtt{P}}_F.\hat{\mathtt{P}}_I.\hat{l}(\hat{k}_f(\eta))]}$$

is smooth-tame. Since the multiplication map

$$\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad (f, g) \mapsto f \cdot g$$

is smooth-tame by Lemma 5.3.6(i) we have shown the tameness of

$$\hat{\delta}_f^\# : \mathtt{O}_f^E \times \mathtt{E}_0 \to \mathtt{E}_1.$$

$\square$

Lemma 7.5.5. *Let $\mu \in \mathbb{C}$ be a complex number with $\mathsf{Re}\,\mu \neq 0$ and define the function*

$$\varphi_\mu : \mathtt{E}_0 \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}), \quad e \mapsto \varphi_\mu.e$$

by

$$(\varphi_\mu.e)(\tau) := \mathsf{Re}(\mu \cdot e(\tau)\,).$$

Then, its inverse is

$$(\varphi_\mu)^{-1} : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}) \to \mathtt{E}_0$$

with

$$(\varphi_\mu)^{-1}(\tau) = \frac{1}{\mu}[2(\hat{\mathtt{P}}_F.\lambda)(\tau) + \frac{\mu}{\mathsf{Re}\,\mu}(\hat{\mathtt{P}}_0.\lambda)(\tau)].$$

Proof. Remember that by Definition 7.3.11 the subspace $\mathtt{E}_0 \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ consists of all functions of the form

$$f(t) = r + a_1 e^{i\tau} + a_2 e^{2i\tau} + \cdots$$

with $r \in \mathbb{R}$ and $a_1, a_2, \ldots \in \mathbb{C}$. By Lemma 7.3.12, we have

$$\mathtt{E}_0 = (\hat{\mathtt{P}}_F + \hat{\mathtt{P}}_R\hat{\mathtt{P}}_0).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$$

and from the fact that the two projections $\hat{\mathtt{P}}_F$ and $\hat{\mathtt{P}}_R\hat{\mathtt{P}}_0$ commutate, we obtain

$$\mathtt{E}_0 = \hat{\mathtt{P}}_F.\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \oplus \hat{\mathtt{P}}_R.\hat{\mathtt{P}}_0.\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}).$$

1. Step:
We define the function

$$\varphi_1 : \hat{\mathtt{P}}_F.\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \hat{\mathtt{P}}_R(1 - \hat{\mathtt{P}}_0).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad e \mapsto \hat{\mathtt{P}}_R.(\mu \cdot e),$$

and will show that

$$\varphi_1^{-1} : \hat{\mathtt{P}}_R(1 - \hat{\mathtt{P}}_0).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \hat{\mathtt{P}}_F.\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \lambda \mapsto \frac{1}{\mu}2\hat{\mathtt{P}}_F.\lambda.$$

is its inverse. The calculation

$$\begin{aligned} \varphi_1.\varphi_1^{-1}.\hat{\mathtt{P}}_R(1 - \hat{\mathtt{P}}_0) &= (\hat{\mathtt{P}}_R.\mu).(\frac{1}{\mu}2\hat{\mathtt{P}}_F).\hat{\mathtt{P}}_R(1 - \hat{\mathtt{P}}_0) \\ &= 2\hat{\mathtt{P}}_R\hat{\mathtt{P}}_F\hat{\mathtt{P}}_R(1 - \hat{\mathtt{P}}_0) \\ &\overset{7.3.3(xiv)}{=} \hat{\mathtt{P}}_R(1 - \hat{\mathtt{P}}_0) \end{aligned}$$

shows that $\varphi_1 \circ \varphi_1^{-1}$ is the identity map on $\hat{\mathtt{P}}_R(1 - \hat{\mathtt{P}}_0).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$, and

$$\varphi_1^{-1}.\varphi_1.\hat{\mathtt{P}}_F = (\frac{1}{\mu}2\hat{\mathtt{P}}_F).(\hat{\mathtt{P}}_R\mu)\hat{\mathtt{P}}_F = \frac{1}{\mu}2\hat{\mathtt{P}}_F\hat{\mathtt{P}}_R\hat{\mathtt{P}}_F\mu \overset{7.3.3(ii)}{=} \frac{1}{\mu}\hat{\mathtt{P}}_F\mu = \hat{\mathtt{P}}_F$$

shows that $\varphi_1^{-1} \circ \varphi_1$ is the identity map on $\hat{\mathtt{P}}_F.\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$.
2. Step:
We define the function

$$\varphi_2 : \hat{\mathtt{P}}_R\hat{\mathtt{P}}_0.\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \hat{\mathtt{P}}_R\hat{\mathtt{P}}_0.\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad e \mapsto \hat{\mathtt{P}}_R.(\mu \cdot e).$$

Its inverse reads

$$\varphi_2^{-1} : \hat{\mathtt{P}}_R\hat{\mathtt{P}}_0.\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \hat{\mathtt{P}}_R\hat{\mathtt{P}}_0.\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \lambda \mapsto \frac{1}{\mathsf{Re}\mu} \cdot \lambda,$$

because $\hat{\mathtt{P}}_R\hat{\mathtt{P}}_0.\mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C})$ is isomorphic to $\mathbb{R}$ as a vector space.
3. Step:
The function

$$\begin{aligned}\varphi_\mu : (\hat{\mathtt{P}}_F.\mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C})) \oplus (\hat{\mathtt{P}}_R\hat{\mathtt{P}}_0.\mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}))\\ \to (\hat{\mathtt{P}}_R(1-\hat{\mathtt{P}}_0).\mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C})) \oplus (\hat{\mathtt{P}}_R\hat{\mathtt{P}}_0.\mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}))\end{aligned}$$

is the direct sum of φ_1 and φ_2, and its inverse is

$$(\varphi_\mu)^{-1} = (\varphi_1)^{-1} \oplus (\varphi_2)^{-1}.$$

Hence, we conclude

$$\begin{aligned}(\varphi_\mu)^{-1}.\lambda &= \frac{1}{\mu}2\hat{\mathtt{P}}_F.\hat{\mathtt{P}}_R(1-\hat{\mathtt{P}}_0).\lambda + \frac{1}{\mathsf{Re}\mu}\hat{\mathtt{P}}_R\hat{\mathtt{P}}_0.\lambda\\ &= \frac{1}{\mu}[2\hat{\mathtt{P}}_F(1-\hat{\mathtt{P}}_0) + \frac{\mu}{\mathsf{Re}\,\mu}\hat{\mathtt{P}}_0].\hat{\mathtt{P}}_R.\lambda\\ &= \frac{1}{\mu}[2\hat{\mathtt{P}}_F + \frac{\mu}{\mathsf{Re}\,\mu}\hat{\mathtt{P}}_0].\hat{\mathtt{P}}_R.\lambda.\end{aligned}$$

□

Lemma 7.5.6. *For $f \in \mathtt{V}$ the inverse map of*

$$\hat{\vartheta}_f : \mathtt{O}_f^E \times \mathtt{E}_0 \to \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{R}),$$

$$\hat{\vartheta}_f(\eta,e)(\tau) := \mathsf{Re}(\hat{\mathtt{m}}(\hat{k}_f(\eta)) \cdot e(\tau))$$

with respect to the right argument reads

$$\hat{\vartheta}_f^\# : \mathtt{O}_f^E \times \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{R}) \to \mathtt{E}_0, \quad (\eta,\lambda) \mapsto \hat{\vartheta}_f^\#(\eta,\lambda),$$

$$\hat{\vartheta}_f^\#(\eta,\lambda)(\tau) = \frac{1}{\hat{\mathtt{m}}(\hat{k}_f(\eta))} \cdot 2(\hat{\mathtt{P}}_F.\lambda)(\tau) + \frac{1}{\mathsf{Re}\,\hat{\mathtt{m}}(\hat{k}_f(\eta))} \cdot (\hat{\mathtt{P}}_0.\lambda)(\tau)$$

and is smooth-tame.

Proof. We substitute $\mu = \hat{\mathtt{m}}(\hat{k}_f(\eta))$ into the statement of Lemma 7.5.5. So, it remains to show the tameness. The three projections

$$\hat{\mathtt{P}}_F, \hat{\mathtt{P}}_I, \hat{\mathtt{P}}_K : \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C})$$

are linear-tame by Lemma 7.3.5. The function

$$\mathtt{O}_f^E \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \eta \to \hat{\mathtt{m}}(\hat{k}_f(\eta))$$

is smooth-tame, because

$$\hat{\mathtt{m}} : \mathtt{W} \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$$

is smooth-tame by Lemma 7.4.9 and

$$\hat{k}_f : \mathtt{U}_f \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathtt{A}_f) \to \mathtt{W}$$

is smooth-tame by Lemma 7.2.10. After several applications of Lemma 5.3.6, we conclude that the function

$$\hat{\vartheta}_f^\# : \mathtt{O}_f^E \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}) \to \mathtt{E}_0, \quad (\eta, \lambda) \mapsto 2\frac{\hat{\mathtt{P}}_F.\lambda}{\hat{\mathtt{m}}(\hat{k}_f(\eta))} + 2\frac{\hat{\mathtt{P}}_0.\lambda}{\hat{\mathtt{P}}_R.\hat{\mathtt{m}}(\hat{k}_f(\eta))}$$

is smooth-tame. □

Remark: The restriction in Section 7.4 to the smaller neighborhood $\mathtt{O}_f^E$ was necessary, in order to get an inverse of $\hat{\vartheta}_f$. In particular, the restriction guarantees that $\mathsf{Re}\,\hat{\mathtt{m}}(\hat{k}_f(\eta))$ is non-zero for $\eta \in \mathtt{O}_f^E$. Otherwise, a division by zero would occur in the proof above.

Lemma 7.5.7. *Let $f \in \mathtt{V}$. Then the inverse map of*

$$\hat{\sigma}_f : \mathtt{O}_f^E \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}),$$

$$\hat{\sigma}_f(\eta, \lambda)(t) := e^{[\hat{\mathtt{P}}_K \hat{l}(\hat{k}_f(\eta))](\tau)} \cdot \lambda(\tau)\big|_{\tau = [\hat{\Sigma}_f(\eta)]^{-1}(t)}$$

with respect to the right argument reads

$$\hat{\sigma}_f^\# : \mathtt{O}_f^E \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}),$$

$$\hat{\sigma}_f^\#(\eta, \lambda)(\tau) = e^{-[\hat{\mathtt{P}}_K.\hat{l}(\hat{k}_f(\eta))](\tau)} \cdot \lambda(\hat{\Sigma}_f(\eta)(\tau)\,)$$

and is smooth-tame.

Proof. By Lemma 7.3.3(xi), the image of $\hat{\mathtt{P}}_K$ consists of real valued functions, such that even

$$-[\hat{\mathtt{P}}_K.\hat{l}(\hat{k}_f(\eta))](\tau)$$

is a real valued. This guarantees that the domains are correct. Now we will show the tameness. The function

$$\hat{\Sigma}_f : \mathtt{U}_f \to \mathsf{Diff}^+\mathbb{S}^1$$

is smooth-tame by Lemma 7.1.6 and the inversion map

$$\iota : \mathsf{Diff}^+\mathbb{S}^1 \to \mathsf{Diff}^+\mathbb{S}^1$$

is smooth-tame by Lemma 5.2.13(ii), and therefore

$$\mathtt{U}_f \to \mathsf{Diff}^+\mathbb{S}^1, \quad \eta \mapsto \iota(\hat{\Sigma}_f(\eta))$$

is smooth-tame. The composition map

$$\mathtt{C} : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \times \mathsf{Diff}^+\mathbb{S}^1 \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad (f, \gamma) \mapsto f \circ \gamma$$

is smooth-tame by Lemma 5.2.12, which implies that the map

$$\mathtt{O}_f^E \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad (\eta, \lambda) \mapsto \lambda \circ \iota(\hat{\Sigma}_f(\eta))$$

is smooth-tame if we take $\mathtt{O}_f^E \subseteq \mathtt{U}_f$ into account. By Lemma 7.5.3(i), the map

$$\mathtt{O}_f^E \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \eta \mapsto e^{-\hat{\mathtt{P}}_K . \hat{l}(\hat{k}_f(\eta))}$$

is smooth-tame, and we conclude that

$$\hat{\sigma}_f^\# : \mathtt{O}_f^E \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}), \quad (\eta, \lambda) \mapsto e^{-[\hat{\mathtt{P}}_K . \hat{l}(\hat{k}_f(\eta))]} \cdot \lambda \circ \iota(\hat{\Sigma}_f(\eta))$$

is smooth-tame, because the mutiplication map is smooth-tame by Lemma 5.3.6(i). $\square$

Proposition 7.5.8 (Inverse of $d\hat{\theta}_f$)**.** *Let $f \in \mathtt{V}$. The derivative of*

$$\hat{\theta}_f : \mathtt{O}_f^E \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$$

is

$$d\hat{\theta}_f : \mathtt{O}_f^E \times \mathtt{E}_1 \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$$

with

$$[d\hat{\theta}_f]_\eta . \lambda = \hat{\sigma}_f(\eta, \hat{\vartheta}_f(\eta, \hat{\delta}_f(\eta, \lambda))).$$

Moreover, the inverse of $d\hat{\theta}_f$ with respect to the right argument reads

$$d[\hat{\theta}_f]^\# : \mathtt{O}_f^E \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}) \to \mathtt{E}_1, \quad (\eta, \Delta h) \mapsto ([d\hat{\theta}_f]_\eta)^{-1} . \Delta h,$$

$$([d\hat{\theta}_f]_\eta)^{-1} . \Delta h = \hat{\delta}_f^\#(\eta, \hat{\vartheta}_f^\#(\eta, \hat{\sigma}_f^\#(\eta, \Delta h)))$$

and is smooth-tame.

Proof. Recall that $\hat{\theta}_f$ was defined to be the restriction of $\hat{\Theta}_f$ to $\mathtt{V}^E \cap \mathtt{U}_f$ in 7.1.4. Here, we restrict $\hat{\theta}_f$ for a further time to $\mathtt{O}_f^E \subseteq \mathtt{V}^E \cap \mathtt{U}_f$. By Lemma 7.5.2, we have

$$d\hat{\theta}_f(\eta, \lambda) = \hat{\sigma}_f(\eta, \hat{\vartheta}_f(\eta, \hat{\delta}_f(\eta, \lambda)))$$

for $\eta \in \mathtt{O}_f^E \subseteq \mathtt{U}_f$ and $\lambda \in \mathtt{E}_0 \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$. This shows that $d\hat{\theta}_f$ is a composition of the three maps

$$\begin{aligned} \hat{\delta}_f &: \mathtt{O}_f^E \times \mathtt{E}_1 \to \mathtt{E}_0, \\ \hat{\vartheta}_f &: \mathtt{O}_f^E \times \mathtt{E}_0 \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}), \quad \text{and} \\ \hat{\sigma}_f &: \mathtt{O}_f^E \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}). \end{aligned}$$

These three maps are discussed in Lemma 7.5.4, Lemma 7.5.6, and Lemma 7.5.7 one by one, such that the rest of this proof is a corollary of these three lemmatas. □

7.6 Theorem

In this section, we collect the work of the preceding sections of this chapter, in order to show that the composition map $\mathtt{C} : \mathtt{V}^E \times \mathsf{Diff}^+\mathbb{S}^1 \to \mathtt{V}$ is a tame diffeomorphism. This already is the main result of this chapter.

Proposition 7.6.1. *If $f \in \mathtt{V}$, then the map*

$$\hat{\theta}_f : \mathtt{O}_f^E \to \hat{\theta}_f(\mathtt{O}_f^E)$$

is a tame diffeomorphism. In particular, $\hat{\theta}_f(\mathtt{O}_f^E) \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$ is an open subset.

Proof. By Lemma 7.4.12, the subset $\mathtt{O}_f^E$ is an open subset of the tame space $\mathtt{E}_1$. By Lemma 7.1.9, the function

$$\hat{\theta}_f : \mathtt{O}_f^E \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$$

is injective, because $\mathtt{O}_f^E \subseteq \mathtt{U}_f \cap \mathtt{V}^E$. Moreover, Lemma 7.1.6 yields that the map $\hat{\theta}_f : \mathtt{O}_f^E \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$ is smooth-tame, because $\mathtt{O}_f^E \subseteq \mathtt{U}_f$. Let

$$d\hat{\theta}_f : \mathtt{O}_f^E \times \mathtt{E}_1 \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$$

the derivative of $\hat{\theta}_f$. By Proposition 7.5.8, its inverse with respect to the right argument is smooth-tame. Finally, we can apply the Nash-Moser Theorem 5.2.15 and obtain that

$$\hat{\theta}_f : \mathtt{O}_f^E \to \hat{\theta}_f(\mathtt{O}_f^E) \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$$

is a tame diffeomorphism. □

Lemma 7.6.2. *Consider the function $\hat{\Theta}_f : \mathtt{U}_f \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$, and its restriction $\hat{\theta}_f : \mathtt{V}^E \cap \mathtt{U}_f \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$. If $f \in \mathtt{V}$, then*

$$\mathtt{O}_f = [\hat{\Theta}_f]^{-1}([\hat{\theta}_f](\mathtt{O}_f^E)).$$

Proof. Recall that Definition 7.4.4 reads

$$\begin{aligned} \mathtt{O}_f &:= \{\, \eta \in \mathtt{V} \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \,:\, \mathsf{Re}(\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(\eta)))) \neq 0 \wedge \eta \in \mathtt{U}_f \,\} \\ \mathtt{O}_f^E &:= \{\, \eta \in \mathtt{V}^E \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \,:\, \mathsf{Re}(\hat{\mathtt{m}}(\hat{k}_f(\eta))) \neq 0 \wedge \eta \in \mathtt{U}_f \,\}. \end{aligned}$$

1. Step: $\eta \in \mathtt{O}_f \Longrightarrow \hat{\Lambda}(\eta) \in \mathtt{O}_f^E$:
Let $\eta \in \mathtt{O}_f$ then $\mathsf{Re}(\hat{\mathtt{m}}(\hat{k}_f(\ \hat{\Lambda}(\eta)\ ))) \neq 0$. Moreover, by Definition 4.2.2 we have

$$\hat{\Lambda}(\eta) \in \mathtt{V}^E.$$

Finally, since $\mathtt{U}_f$ is $\mathsf{Diff}^+\mathbb{S}^1$-invariant, we have $\hat{\Lambda}(\eta) = \eta \circ \hat{\Gamma}(\eta)^{-1} \in \mathtt{U}_f$, and conclude $\hat{\Lambda}(\eta) \in \mathtt{O}_f^E$. Particularly, we have $\hat{\Lambda}(\eta) \in \mathtt{V}^E \cap \mathtt{U}_f$.
2. Step: $\mathtt{O}_f \subseteq [\hat{\Theta}_f]^{-1}([\hat{\theta}_f](\mathtt{O}_f^E))$:
For $\eta \in \mathtt{O}_f$ we have

$$\hat{\Theta}_f(\eta) \overset{7.1.3(i)}{=} \hat{\Theta}_f(\eta \circ \hat{\Gamma}(\eta)^{-1}) \overset{4.2.3(i)}{=} \hat{\Theta}_f(\hat{\Lambda}(\eta)) \overset{7.1.4}{=} \hat{\theta}_f(\hat{\Lambda}(\eta)),$$

and get

$$\eta \in [\hat{\Theta}_f]^{-1}([\hat{\theta}_f](\hat{\Lambda}(\eta))) \subseteq [\hat{\Theta}_f]^{-1}([\hat{\theta}_f](\mathtt{O}_f^E)).$$

3. Step: $[\hat{\Theta}_f]^{-1}([\hat{\theta}_f](\mathtt{O}_f^E)) \subseteq \mathtt{O}_f$:
Let $\eta \in [\hat{\Theta}_f]^{-1}([\hat{\theta}_f](\mathtt{O}_f^E))$, then there exists an element $\lambda \in \mathtt{O}_f^E$ such that $\hat{\Theta}_f(\eta) = \hat{\Theta}_f(\lambda)$, and Lemma 7.1.10 yields

$$\hat{\Lambda}(\eta) = \hat{\Lambda}(\lambda).$$

Since $\lambda \in \mathtt{O}_f^E \subseteq \mathtt{V}^E$, we have $\hat{\Lambda}(\lambda) = \lambda$ by 7.1.3(iii). Hence, $\hat{\Lambda}(\eta) \in \mathtt{O}_f^E$, and we conclude $\eta \in \mathtt{O}_f^E$. □

Remark: We can consider $\mathtt{O}_f$ as the orbit of $\mathtt{O}_f^E$ under the right multiplication by the diffeomorphism group $\mathsf{Diff}^+\mathbb{S}^1$.

Lemma 7.6.3. *Let $f \in \mathtt{V}$, then the subset $\mathtt{O}_f \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is an open neighborhood of f.*

Proof. 1. Step: $\mathtt{O}_f$ is an open subset of $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$:
By Lemma 7.6.1, the map

$$\hat{\theta}_f : \mathtt{O}_f^E \to \hat{\theta}_f(\mathtt{O}_f^E) \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$$

is a tame diffeomorphism. In particular, its image $\hat{\theta}_f(\mathtt{O}_f^E)$ is an open subset of $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$. Since

$$\hat{\Theta}_f : \mathtt{U}_f \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$$

is continuous by Lemma 7.1.6, we conclude that

$$[\hat{\Theta}_f]^{-1}(\hat{\theta}_f(\mathtt{O}_f^E)) \subseteq \mathtt{U}_f$$

is open. By Lemma 7.6.2, we have

$$\mathtt{O}_f = [\hat{\Theta}_f]^{-1}([\hat{\theta}_f](\mathtt{O}_f^E)), \qquad (\dagger)$$

and since $\mathtt{U}_f \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is an open subset by Lemma 7.1.5 and we conclude that even $\mathtt{O}_f$ is an open subset of $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$.
2. Step: $f \in \mathtt{O}_f$:
By Lemma 7.4.15, we have $\hat{\Lambda}(f) \in \mathtt{O}_f^E$, and therefore $\hat{\Theta}_f(\hat{\Lambda}(f)) \in \hat{\Theta}_f(\mathtt{O}_f^E)$. Hence,

$$\hat{\Theta}_f(f) \overset{7.1.7}{=} \hat{\Theta}_f(\hat{\Lambda}(f)) \in \hat{\Theta}_f(\mathtt{O}_f^E)$$

implies

$$f \in [\hat{\Theta}_f]^{-1}(\hat{\Theta}_f(\mathtt{O}_f^E)) \overset{7.1.4}{=} [\hat{\Theta}_f]^{-1}(\hat{\theta}_f(\mathtt{O}_f^E)) \overset{(\dagger)}{=} \mathtt{O}_f.$$

□

Lemma 7.6.4. *The map $\hat{\Lambda} : \mathtt{V} \to \mathtt{V}^E$ is smooth-tame.*

Proof. Let $f \in \mathtt{V}$. By Lemma 7.6.3, the subset $\mathtt{O}_f \subseteq \mathtt{V}$ is an open neighborhood of f. The map

$$\hat{\Theta}_f : \mathtt{O}_f \subseteq \mathtt{U}_f \to \hat{\theta}_f(\mathtt{O}_f^E) \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$$

is smooth-tame by Lemma 7.1.6 and the map

$$[\hat{\theta}_f]^{-1} : \hat{\theta}_f(\mathtt{O}_f^E) \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}) \to \mathtt{O}_f^E$$

is smooth-tame by Lemma 7.6.1. Lemma 7.1.10(i) states

$$\hat{\Lambda} = [\hat{\theta}_f]^{-1} \circ \hat{\Theta}_f$$

and hence $\hat{\Lambda} : \mathtt{O}_f \to \mathtt{O}_f^E$ is smooth-tame. Since $\mathtt{V} = \bigcup_{f \in \mathtt{V}} \mathtt{O}_f$ is an open cover, the lemma is proved. □

Lemma 7.6.5. *The map* $\hat{\Gamma} : \mathtt{V} \to \mathsf{Diff}^+\mathbb{S}^1$ *is smooth-tame.*

Proof. Let $f \in \mathtt{V}$, then by Lemma 7.1.5 the subset $\mathtt{U}_f \subseteq \mathtt{V}$ is an open neighborhood of f. Lemma 7.1.10(ii) asserts that

$$\hat{\Gamma}(\eta) = \iota(\hat{\Sigma}_f(\hat{\Lambda}(\eta))) \circ \hat{\Sigma}_f(\eta)$$

holds for all $\eta \in \mathtt{U}_f$. By Lemma 7.1.2, we have $\hat{\Lambda}(\mathtt{U}_f) \subseteq \mathtt{U}_f$, and by Lemma 7.6.4 we know that the map $\hat{\Lambda} : \mathtt{U}_f \to \mathtt{U}_f$ is smooth-tame. Moreover, by Lemma 7.1.6 the map

$$\hat{\Sigma}_f : \mathtt{U}_f \to \mathsf{Diff}^+\mathbb{S}^1$$

is smooth-tame, and by Lemma 5.2.13 the group inversion

$$\iota : \mathsf{Diff}^+\mathbb{S}^1 \to \mathsf{Diff}^+\mathbb{S}^1$$

is smooth-tame. Hence, $\hat{\Gamma} : \mathtt{U}_f \to \mathsf{Diff}^+\mathbb{S}^1$ is smooth-tame and taking

$$\mathtt{V} = \bigcup_{f \in \mathtt{V}} \mathtt{U}_f$$

into account, we have proved this lemma. $\square$

Theorem 7.6.6. *The composition map*

$$\mathtt{C} : \mathtt{V}^E \times \mathsf{Diff}^+\mathbb{S}^1 \to \mathtt{V}$$

is a tame diffeomorphism and its inverse function reads

$$\mathtt{C}^{-1} : \mathtt{V} \to \mathtt{V}^E \times \mathsf{Diff}^+\mathbb{S}^1, \quad f \mapsto (\hat{\Lambda}(f), \hat{\Gamma}(f)).$$

Proof. The composition map $\mathtt{C} : \mathtt{V}^E \times \mathsf{Diff}^+\mathbb{S}^1 \to \mathtt{V}$ is bijective by Theorem 4.2.1. Furthermore, it is smooth-tame by Lemma 5.2.12. The map

$$\hat{\Lambda}(f) : \mathtt{V} \to \mathtt{V}^E$$

is smooth-tame by Lemma 7.6.4 and the map

$$\hat{\Gamma}(f) : \mathtt{V} \to \mathsf{Diff}^+\mathbb{S}^1$$

is smooth-tame Lemma 7.6.5, hence the inverse $\mathtt{C}^{-1}$ is also smooth-tame. $\square$

Chapter 8

The Complex Structure

Let $\mathsf{Rot}^+\mathbb{S}^1 \subseteq \mathsf{Diff}^+\mathbb{S}^1$ be the subgroup of rotations preserving the metric on the 1-sphere $\mathbb{S}^1$. In this chapter, we will show that on the homogeneous space $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ there exists exactly one complex structure up to sign which is invariant under the action of $\mathsf{Diff}^+\mathbb{S}^1$. In Section 8.1, we draw an atlas of the homogeneous space which consists of the chart

$$\chi : \mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1 \to \mathtt{M},$$

where

$$\mathtt{M} := \{ \lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}) \, : \, \int_{\mathbb{S}^1} \lambda(t)dt = 0 \wedge \lambda'(t) > -1 \}$$

is an open subset of a vector space. It turns out that the induced action

$$\mathsf{Diff}^+\mathbb{S}^1 \times \mathtt{M} \to \mathtt{M}$$

is an affine action. In Section 8.2, we show that there exists at least one almost complex structure (up to sign) on $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ respectively $\mathtt{M}$ which is invariant and integrable. With the results of the preceding chapters, we define in Section 8.3 a bijective map

$$\hat{\Xi} : \mathtt{V}^- \to \mathtt{M}.$$

Recall that $\mathtt{V}^-$ is a complex vector space (Definition 2.4.3). This non-linear function is a tame diffeomorphism and is holomorphic, which will be shown in the rest of this chapter. To reach this goal, we compute the derivative of $\hat{\Xi}$ in Section 8.4. The derivative can be written as a product of three functions (Proposition 8.4.14)

$$d\hat{\Xi} = \hat{\omega}_{\hat{\Gamma}(f)^{-1}} \circ \hat{\mathtt{P}}_K \circ \hat{\mathtt{h}}_f.$$

Section 8.5 is devoted to compute the inverse function of the linear function $\hat{\mathtt{P}}_K$, so that we can show in Section 8.6 that $d\hat{\Xi}^{\#}$ is smooth-tame and can apply the Nash-Moser Theorem. The result is that $\hat{\Xi} : \mathtt{V}^- \to \mathtt{M}$ is a tame diffeomorphism which is biholomorphic, such that $\mathtt{M}$ (and hence $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$) becomes a complex manifold. With the result of section 8.2, it is shown that this complex structure is unique (modulo sign).

8.1 Homogeneous Space

In this section, we build a chart (Lemma 8.1.6) of the homogeneous space

$$\hat{\Psi} : \mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1 \to \mathtt{M}$$

with the open subset

$$\mathtt{M} := \{\, \lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}) \,:\, \int_{\mathbb{S}^1} \lambda(t)dt = 0 \wedge \lambda'(t) > -1 \,\}$$

of the vector space $\mathtt{K}$. The action of $\mathsf{Diff}^+\mathbb{S}^1$ on $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ induces an affine action

$$\hat{\rho} : \mathsf{Diff}^+\mathbb{S}^1 \times \mathtt{M} \to \mathtt{M}, \quad (\gamma, m) \mapsto \hat{\omega}_\gamma.m + \hat{\Pi}(\gamma^{-1}).$$

Definition 8.1.1. We define the subspace

$$\mathtt{K} := \{\, \lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}) \,:\, \hat{\mathtt{P}}_0.\lambda = 0 \,\}.$$

Recall, that Lemma 7.3.6(ii) describes the projection

$$\hat{\mathtt{P}}_0 : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \lambda \mapsto \hat{\mathtt{P}}_0.\lambda$$

by

$$(\hat{\mathtt{P}}_0.\lambda)(\tau) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \lambda(t)dt.$$

Remark: $\mathtt{K} = \hat{\mathtt{P}}_R.(1 - \hat{\mathtt{P}}_0).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$.

Definition 8.1.2. We define the subset

$$\mathtt{M} := \{\, m \in \mathtt{K} \,:\, \forall_{t \in \mathbb{S}^1}\, m'(t) > -1 \,\}.$$

Remark: The subset $\mathtt{M}$ is an open convex subset of the vector space $\mathtt{K}$. In this way $\mathtt{K}$ can be considered as the tangent space of the manifold $\mathtt{M}$.

Lemma 8.1.3. *The function*

$$\hat{\Pi} : \mathsf{Diff}^{+}\mathbb{S}^1 \to \mathtt{M}, \quad \gamma \mapsto \hat{\Pi}(\gamma) \;:=\; (1-\hat{\mathsf{P}}_0).(\gamma - \mathsf{id}_{\mathbb{S}^1}) \;=\; (\gamma - \mathsf{id}_{\mathbb{S}^1}) - \frac{1}{2\pi}\int_{\mathbb{S}^1} (\gamma(t)-t)dt,$$

is surjective.

Proof. 1. Step: $\hat{\Pi}$ is well-defined:
According Definition 2.1.5, we consider an element $\gamma \in \mathsf{Diff}^{+}\mathbb{S}^1$ as an equivalence class of functions $f : \mathbb{R} \to \mathbb{R}$ with the properties

$$f(t+2\pi) = 2\pi + f(t) \qquad \text{and} \qquad f'(t) > 0.$$

Hence, $f - \mathsf{id}_{\mathbb{S}^1}$ is an equivalence class of 2π-periodic functions, whose representatives differ only by a constant. The projection $1-\hat{\mathsf{P}}_0$ determines this constant. In this way, the expression

$$(1-\hat{\mathsf{P}}_0).(\gamma - \mathsf{id}_{\mathbb{S}^1}) \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$$

is well-defined. We have
(i) $\hat{\mathsf{P}}_0.(1-\hat{\mathsf{P}}_0).(\gamma - \mathsf{id}_{\mathbb{S}^1}) \equiv 0,$
(ii) $[(1-\hat{\mathsf{P}}_0).(\gamma - \mathsf{id}_{\mathbb{S}^1})]'(t) = \gamma'(t) - 1 > -1,$
and therefore $(1-\hat{\mathsf{P}}_0).(\gamma - \mathsf{id}_{\mathbb{S}^1}) \in \mathtt{M}$.
2. Step:
To show that $\hat{\Pi} : \mathsf{Diff}^{+}\mathbb{S}^1 \to \mathtt{M}$ is surjective, we construct a preimage for $\lambda \in M$. We have $\lambda'(t) < -1$ and $\lambda : \mathbb{R} \to \mathbb{R}$ is a smooth 2π-periodic function. Therefore, the function

$$f : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \lambda(t) + t$$

fulfills $f \in \widetilde{\mathsf{Diff}}^{+}\mathbb{S}^1$, and hence its equivalence class $\gamma = [f]$ is the preimage of λ. □

Lemma 8.1.4. *The derivative of* $\hat{\Pi}$ *is*

$$d\hat{\Pi} : \mathsf{Diff}^{+}\mathbb{S}^1 \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}) \to \mathtt{K} \text{ with } d\hat{\Pi}(\gamma, \Delta\gamma) = (1-\hat{\mathsf{P}}_0).\Delta\gamma.$$

Proof.

$$\begin{aligned} & \hat{\Pi}(\gamma + \Delta\gamma) - \hat{\Pi}(\gamma) \\ = \;& (1-\hat{\mathsf{P}}_0)(\gamma + \Delta\gamma - \mathsf{id}_{\mathbb{S}^1}) - (1-\hat{\mathsf{P}}_0)(\gamma - \mathsf{id}_{\mathbb{S}^1}) \\ = \;& (1-\hat{\mathsf{P}}_0).\Delta\gamma. \end{aligned}$$

□

Lemma 8.1.5. *Let* $\gamma_1, \gamma_2 \in \mathsf{Diff}^+\mathbb{S}^1$*, then the following assertions are equivalent:*
(i) $\hat{\Pi}(\gamma_1) = \hat{\Pi}(\gamma_2)$.
(ii) $\gamma_1 \circ \gamma_2^{-1} \in \mathsf{Rot}^+\mathbb{S}^1$.

Proof. (i)$\Longrightarrow$(ii): The definition of $\hat{\Pi}$ yields

$$(\gamma_1 - \mathsf{id}_{\mathbb{S}^1}) - \hat{\mathtt{P}}_0.(\gamma_1 - \mathsf{id}_{\mathbb{S}^1}) = (\gamma_2 - \mathsf{id}_{\mathbb{S}^1}) - \hat{\mathtt{P}}_0.(\gamma_2 - \mathsf{id}_{\mathbb{S}^1}).$$

Let $\xi(t) = t + \hat{\mathtt{P}}_0.(\gamma_1 - \mathsf{id}_{\mathbb{S}^1}) - \hat{\mathtt{P}}_0.(\gamma_2 - \mathsf{id}_{\mathbb{S}^1})$ then we have

$$\gamma_1(t) = \xi(\gamma_2(t)).$$

Hence, $\gamma_1 \circ \gamma_2^{-1} = \xi \in \mathsf{Rot}^+\mathbb{S}^1$.
(ii)$\Longrightarrow$(i): We have to show that

$$\hat{\Pi}(\gamma) = \hat{\Pi}(\xi \circ \gamma)$$

for $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$ and $\xi \in \mathsf{Rot}^+\mathbb{S}^1$. Since $\xi \in \mathsf{Rot}^+\mathbb{S}^1$ there exists a real constant $C \in \mathbb{R}$ such that $\xi : \mathbb{S}^1 \to \mathbb{S}^1,\ t \mapsto t + C$. Let $f : \mathbb{R} \to \mathbb{R}$ be a representative for $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$ according Definition 2.1.5, then

$$g : \mathbb{R} \to \mathbb{R}, \quad t \mapsto f(t) + C$$

is a representative of $\xi \circ \gamma$. Since $g - \mathsf{id}_{\mathbb{S}^1} = f - \mathsf{id}_{\mathbb{S}^1} + C$ we have

$$\hat{\Pi}(g) = (1 - \hat{\mathtt{P}}_0).(g - \mathsf{id}_{\mathbb{S}^1}) = (1 - \hat{\mathtt{P}}_0).(f - \mathsf{id}_{\mathbb{S}^1}) + \underbrace{(1 - \hat{\mathtt{P}}_0).C}_{=0} = \hat{\Pi}(f).$$

□

Lemma 8.1.6. *The mapping*

$$\hat{\Psi} : \mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1 \to \mathtt{M}, \quad \gamma \circ \mathsf{Rot}^+\mathbb{S}^1 \mapsto \hat{\Pi}(\gamma^{-1})$$

is well-defined and bijective.

Proof. 1. Step: $\hat{\Psi}$ is well-defined:
Let $[\gamma] = \gamma \circ \mathsf{Rot}^+\mathbb{S}^1 \in \mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ and γ_1, γ_2 be two representatives of $[\gamma]$, i.e., $\xi = \gamma_1^{-1} \circ \gamma_2 \in \mathsf{Rot}^+\mathbb{S}^1$. Then Lemma 8.1.5 yields

$$\hat{\Pi}(\gamma_1^{-1}) = \hat{\Pi}(\xi^{-1} \circ \gamma_2^{-1}) = \hat{\Pi}(\gamma_2^{-1}).$$

2. Step: $\hat{\Psi}$ is surjective:
Since $\hat{\Pi} : \mathsf{Diff}^+\mathbb{S}^1 \to \mathtt{M}$ is surjective by Lemma 8.1.3 the mapping $\hat{\Psi}$ is also surjective.
3. Step: $\hat{\Psi}$ is injective:
Let $[\gamma_1], [\gamma_2] \in \mathsf{Diff}^+\mathbb{S}^1$ with $\hat{\Psi}([\gamma_1]) = \hat{\Psi}([\gamma_2])$. Then $\hat{\Pi}(\gamma_1^{-1}) = \hat{\Pi}(\gamma_2^{-1})$ and by Lemma 8.1.5 we have $\gamma_1^{-1} \circ \gamma_2 \in \mathsf{Rot}^+\mathbb{S}^1$. Hence, $[\gamma_1] = [\gamma_2]$. □

Definition 8.1.7 (Action of $\mathsf{Diff}^+\mathbb{S}^1$ on $\mathtt{K}$)**.** We define the action

$$\hat{\omega} : \mathsf{Diff}^+\mathbb{S}^1 \times \mathtt{K} \to \mathtt{K}, \quad (\gamma, b) \mapsto \hat{\omega}_\gamma.b$$

by

$$\hat{\omega}_\gamma.b := (1 - \hat{\mathsf{P}}_0).\mathsf{R}_{\gamma^{-1}}.b. = (1 - \hat{\mathsf{P}}_0).(b \circ \gamma^{-1}).$$

Lemma 8.1.8 (Cocycle properties of $\hat{\Pi}$)**.** *We have*
(i) $\hat{\Pi}(\mathsf{id}_{\mathbb{S}^1}) = 0$, *and*
(ii) $\hat{\Pi}(\xi \circ \gamma) = \hat{\omega}_{\gamma^{-1}}.\hat{\Pi}(\xi) + \hat{\Pi}(\gamma)$ *for* $\xi, \gamma \in \mathsf{Diff}^+\mathbb{S}^1$.

Proof. (i): $\hat{\Pi}(\mathsf{id}_{\mathbb{S}^1}) = (1 - \hat{\mathsf{P}}_0).(\mathsf{id}_{\mathbb{S}^1} - \mathsf{id}_{\mathbb{S}^1}) = 0$.
(ii):

$$\begin{aligned}
\hat{\Pi}(\xi \circ \gamma) &= (1 - \hat{\mathsf{P}}_0).(\xi \circ \gamma - \mathsf{id}_{\mathbb{S}^1}) \\
&= (1 - \hat{\mathsf{P}}_0).(\xi \circ \gamma - \gamma) + (1 - \hat{\mathsf{P}}_0).(\gamma - \mathsf{id}_{\mathbb{S}^1}) \\
&= (1 - \hat{\mathsf{P}}_0).[(\xi \circ \gamma - \gamma) + \hat{\mathsf{P}}_0.(\xi - \mathsf{id}_{\mathbb{S}^1})] + \hat{\Pi}(\gamma) \\
&= (1 - \hat{\mathsf{P}}_0).[\mathsf{R}_\gamma.(\xi - \mathsf{id}_{\mathbb{S}^1}) + \mathsf{R}_\gamma.\hat{\mathsf{P}}_0.(\xi - \mathsf{id}_{\mathbb{S}^1})] + \hat{\Pi}(\gamma) \\
&= (1 - \hat{\mathsf{P}}_0).\mathsf{R}_\gamma.(1 - \hat{\mathsf{P}}_0).(\xi - \mathsf{id}_{\mathbb{S}^1}) + \hat{\Pi}(\gamma) \\
&= \hat{\omega}_{\gamma^{-1}}.\hat{\Pi}(\xi) + \hat{\Pi}(\gamma).
\end{aligned}$$

□

Lemma 8.1.9 (Action of $\mathsf{Diff}^+\mathbb{S}^1$ on $\mathtt{M}$)**.** *The bijective function* $\hat{\Psi} : \mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1 \to \mathtt{M}$ *induces the affine action*

$$\hat{\rho} : \mathsf{Diff}^+\mathbb{S}^1 \times \mathtt{M} \to \mathtt{M}, \quad (\gamma, m) \mapsto \hat{\rho}_\gamma.m$$

defined by

$$\begin{aligned}
\hat{\rho}_\gamma(m) &:= \hat{\omega}_\gamma.m + \hat{\Pi}(\gamma^{-1}) \\
&= (1 - \hat{\mathsf{P}}_0).(m \circ \gamma^{-1}) + (1 - \hat{\mathsf{P}}_0).(\gamma^{-1} - \mathsf{id}_{\mathbb{S}^1}).
\end{aligned}$$

Proof. The action of $\mathsf{Diff}^+\mathbb{S}^1$ on $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ is just the left action

$$\gamma \circ [\eta] = [\gamma \circ \eta].$$

So, we need to show that

$$\hat{\Psi}(\gamma.[\eta]) = \rho_\gamma \hat{\Psi}([\eta])$$

holds for $\gamma, \eta \in \mathsf{Diff}^+\mathbb{S}^1$.

$$\begin{aligned}\hat{\Psi}(\gamma.[\eta]) &= \hat{\Psi}([\gamma \circ \eta]) \\ &= \hat{\Pi}(\eta^{-1} \circ \gamma^{-1}) \\ &\overset{8.1.8(ii)}{=} \hat{\omega}_\gamma.\hat{\Pi}(\eta^{-1}) + \hat{\Pi}(\gamma^{-1}) \\ &\overset{8.1.9}{=} \hat{\rho}_\gamma(\hat{\Pi}(\eta^{-1})) \\ &= \hat{\rho}_\gamma(\hat{\Psi}([\eta])).\end{aligned}$$

□

The following lemma confirms that the derivative of the affine action $\hat{\rho}_\gamma$ is the linear part $\hat{\omega}_\gamma$.

Lemma 8.1.10 (Derivative of $\hat{\rho}$)**.** *Let $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$. Then the derivative of the map $\hat{\rho}_\gamma : \mathtt{M} \longrightarrow \mathtt{M}$ at the point $m \in \mathtt{M}$ is*

$$d\hat{\rho}_\gamma : \mathtt{M} \times \mathtt{K} \longrightarrow \mathtt{K}, \quad (m, \Delta m) \mapsto d\hat{\rho}_\gamma(m, \Delta m) = \hat{\omega}_\gamma.\Delta m.$$

Proof. Consider

$$\hat{\rho}_\gamma : \mathtt{M} \longrightarrow \mathtt{M}, \quad m \longrightarrow \hat{\omega}_\gamma.m + \hat{\Pi}(\gamma^{-1}).$$

The term $\hat{\Pi}(\gamma^{-1})$ can be treated as a constant, and

$$m \mapsto \hat{\omega}_\gamma.m = (1 - \hat{\mathsf{P}}_0).\mathsf{R}_{\gamma^{-1}}.m$$

is a linear mapping. □

8.2 Almost Complex Structure

Consider a manifold M. An almost complex structure is a function

$$J : T(M) \to T(M)$$

on the tangent bundle such that $J_m := J|_{T_m(M)}$ is a linear map with $J_m^2 = -1$ for all $m \in M$. This defines by no means a complex manifold. For a complex manifold a holomorphic atlas is necessary. In this section, we show that there exists at least one almost complex structure on $\mathtt{M}$ which is invariant under $\mathsf{Diff}^+\mathbb{S}^1$, and integrable. Be cautious, even since $\mathtt{M}$ is an open subset of a complex vector space, the complex structure of the tangent space changes from point to point.

Definition 8.2.1 (Complex structure on a vector space)**.** Let V be a real vector space. A linear function $j : V \to V$ is called a *complex structure* on V if $j^2 = -1$.

Definition 8.2.2 (Almost complex structure)**.** Let M be a real manifold. A function

$$J : T(M) \to T(M)$$

is called an *almost complex structure* if
(i) $J_m := J|_{T_m(M)} : T_m(M) \to T_m(M)$ is a linear map, and
(ii) $J_m^2 = -1$ for all $m \in M$.

Example Every complex structure on a real vector space V provides V with an almost complex structure regarding V as a manifold.

Definition 8.2.3 (Holomorphic function)**.** Let (M, J_M) and (N, J_N) be two real manifolds endowed with almost complex structures. A continuous differentiable function

$$\varphi : M \to N$$

is called *holomorphic* if

$$d\varphi \circ J_M = J_N \circ d\varphi.$$

Definition 8.2.4 (Integrable almost complex structure)**.** Let M be a real manifold with an almost complex structure

$$J : T(M) \to T(M).$$

Split the complexified tangent bundle $T(M)_{\mathbb{C}} = T(M) \otimes \mathbb{C}$ into two subbundles

$$T(M)_{\mathbb{C}} = T^+(M) \oplus T^-(M)$$

with

$$\begin{aligned} T^+(M) &:= \bigcup_{m \in M} \{\, w \in T_m(M)_{\mathbb{C}} \;:\; J_m.w = iw \,\}, \text{ and} \\ T^-(M) &:= \bigcup_{m \in M} \{\, w \in T_m(M)_{\mathbb{C}} \;:\; J_m.w = -iw \,\}. \end{aligned}$$

The almost complex structure $J : T(M) \to T(M)$ is an *integrable almost complex structure* if $T^+(M)$ is integrable as a distribution. This means that

$$[\xi_1, \xi_2] \in \Gamma(T^+(M))$$

for $\xi_1, \xi_2 \in \Gamma(T^+(M))$, where $\Gamma(T^+(M))$ is the set of vector fields which are sections in the subbundle $T^+(M)$. Moreover, the vector fields of $\Gamma(T^+(M))$ are called *holomorphic vector fields* with respect to J.

Definition 8.2.5 (Complex structure on a manifold)**.** Let M be a real manifold. An almost complex structure

$$J : T(M) \to T(M)$$

is a *complex structure* if there exists a complex atlas. This means for every point $m \in M$ there exists a neighborhood U_m and a biholomorphic function

$$\phi : U_m \to W$$

into an open subset W of a complex vector space. In this case J turns M into a complex manifold.

Lemma 8.2.6. *Every complex structure is an integrable almost complex structure.*

Proof. On a complex vector space, the holomorphic vector fields are closed under the commutator bracket. □

Remark:

In the finite dimensional case, the Newlander Nirenberg Theorem [32, 24] asserts that an integrable almost complex structure is a complex structure. For Fréchet-manifolds this theorem fails.

Definition 8.2.7 (Complex structure on $\mathtt{K}$)**.** Let $\sigma : \mathbb{N} \to \{-1, 1\}$ be a function. Define a complex structure on the vector space $\mathtt{K}$ (Definition 8.1.1) by

$$j_\sigma : \mathtt{K} \to \mathtt{K}, \quad \lambda \to j_\sigma.\lambda$$

with

$$(j_\sigma.\lambda)(t) = \sum_{k=1}^{\infty} (i)^{\sigma(k)} a_k e^{ikt} + (-i)^{\sigma(k)} \bar{a}_k e^{-ikt}$$

for

$$\lambda(t) = \sum_{k=1}^{\infty} a_k e^{ikt} + \bar{a}_k e^{-ikt}.$$

Properties:

(i) j_σ is invariant under $\mathsf{Rot}^+\mathbb{S}^1$, i.e., $j_\sigma \circ \hat{\omega}_\xi = \hat{\omega}_\xi \circ j_\sigma$ for all $\xi \in \mathsf{Rot}^+\mathbb{S}^1$.

(ii) If $\sigma \equiv 1$, then $j_\sigma = \hat{\mathtt{H}}$ due to Lemma 7.3.7.

Definition 8.2.8. Let $\sigma : \mathbb{N} \to \{-1, 1\}$ be a function. On the manifold $\mathtt{M}$, we define an almost complex structure

$$J_\sigma : \mathtt{M} \times \mathtt{K} \to \mathtt{K}, \quad (m, b) \mapsto (J_\sigma)_m.b$$

by

$$(J_\sigma)_m = \hat{\omega}_{\gamma^{-1}} \circ j_\sigma \circ \hat{\omega}_\gamma,$$

where we have chosen $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$ such that $\hat{\Pi}(\gamma) = m$.

Remark: We have to show that this definition is independent of the choice of γ. Let $\gamma_1, \gamma_2 \in \mathsf{Diff}^+\mathbb{S}^1$ with $\hat{\Pi}(\gamma_1) = \hat{\Pi}(\gamma_2)$, then $\xi := \gamma_1 \circ \gamma_2^{-1} \in \mathsf{Rot}^+\mathbb{S}^1$ due to Lemma 8.1.5. This implies

$$\begin{aligned}
(J_\sigma)_{\hat{\Pi}(\gamma_1)} &= \hat{\omega}_{\gamma_1^{-1}} \circ j_\sigma \circ \hat{\omega}_{\gamma_1} \\
&= \hat{\omega}_{\gamma_2^{-1}} \circ \hat{\omega}_{\xi^{-1}} \circ j_\sigma \circ \hat{\omega}_\xi \circ \hat{\omega}_{\gamma_2} \\
&= \hat{\omega}_{\gamma_2^{-1}} \circ j_\sigma \circ \hat{\omega}_{\gamma_2} \\
&= (J_\sigma)_{\hat{\Pi}(\gamma_2)}.
\end{aligned}$$

Lemma 8.2.9. *Every complex structure $j : \mathtt{K} \to \mathtt{K}$ on the vector space $\mathtt{K}$ which is invariant under $\mathsf{Rot}^+\mathbb{S}^1$ is of the form of Definition 8.2.7 for a given $\sigma : \mathbb{N} \to \{-1, 1\}$. The invariance reads*

$$j \circ \hat{\omega}_\xi = \hat{\omega}_\xi \circ j \qquad \text{for all } \xi \in \mathsf{Rot}^+\mathbb{S}^1$$

in a formula.

Proof. Let $j : \mathtt{K} \to \mathtt{K}$ be a complex structure on the vector space V, and extend it to a complex linear map on $\mathtt{K}_\mathbb{C}$. Let $b_k(t) = e^{ikt}$ and $V_k := \mathbb{C} \cdot b_k$ for $k \in \mathbb{Z}\backslash\{0\}$. Then

$$\mathtt{K}_\mathbb{C} = \overline{\bigoplus_{k \in \mathbb{Z}\backslash\{0\}} V_k}$$

where $\mathtt{K}_\mathbb{C} = \mathtt{K} \otimes \mathbb{C}$. Let $\xi \in \mathsf{Rot}^+\mathbb{S}^1$ defined by

$$\xi : \mathbb{S}^1 \to \mathbb{S}^1, \quad t \mapsto t + 2\pi\sqrt{2},$$

then $\hat{\omega}_\xi . b_k = \lambda_k b_k$ with $\lambda_k = e^{i2\pi k\sqrt{2}}$. Moreover, all eigenvalues λ_k are different for different indices $k \in \mathbb{Z}\backslash\{0\}$. The family $\{V_k\}_{k \in \mathbb{Z}\backslash\{0\}}$ consists of the eigenspaces of $\hat{\omega}_\xi$. The fact that j is invariant under $\mathsf{Rot}^+\mathbb{S}^1$ implies

$$[\omega_\xi, j] = 0,$$

and therefore V_k is a j-invariant subspace. Hence, b_k is an eigenvector of j, because V_k is a one dimensional subspace. Now we are able to reconstruct $\sigma : \mathbb{N} \to \{-1, 1\}$. For $k \in \mathbb{N}$ define $\sigma(k)$ by

$$j.b_k = (i)^{\sigma(k)}.b_k$$

keeping in mind that the only possible eigenvalues of the complex structure j are i and $-i$. Since j maps $\mathtt{K}$ to $\mathtt{K}$, we have

$$\overline{j.b_k} = j.\bar{b}_k,$$

and we can compute

$$j.b_{-k} = j.\bar{b}_k = \overline{j.b_k} = \overline{(i)^{\sigma(k)}.b_k} = (-i)^{\sigma(k)} b_{-k}.$$

This proves that $j = j_\sigma$ for the $\sigma : \mathbb{N} \to \{-1, 1\}$ above (Definition 8.2.7). $\square$

Lemma 8.2.10. *Let $J : T(\mathtt{M}) \to T(\mathtt{M})$ be an almost complex structure on* $\mathtt{M}$*. Then the following three statements are equivalent:*

(i) *J is invariant under the action*

$$\hat{\rho} : \mathsf{Diff}^+\mathbb{S}^1 \times \mathtt{M} \to \mathtt{M}, \quad (\gamma, m) \mapsto \hat{\rho}_\gamma.m,$$

i.e.,

$$J_{\hat{\rho}_\gamma(m)}.d\rho_\gamma(m, \Delta m) = d\hat{\rho}_\gamma(m, J_m.\Delta m)$$

for all $m \in \mathtt{M}$, $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$, and $\Delta m \in \mathtt{K}$.

(ii) $J_{\hat{\Pi}(\gamma)} = \hat{\omega}_{\gamma^{-1}} \circ J_0 \circ \hat{\omega}_\gamma$ *for all $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$.*

(iii) $J_{\hat{\Pi}(\gamma_1 \circ \gamma)} = \hat{\omega}_{\gamma^{-1}} \circ J_{\hat{\Pi}(\gamma_1)} \circ \hat{\omega}_\gamma$ *for all $\gamma, \gamma_1 \in \mathsf{Diff}^+\mathbb{S}^1$.*

Proof. (i)$\Longleftrightarrow$(iii):
Due to Lemma 8.1.10 the statement

$$J_{\hat{\rho}_\gamma(m)}.d\rho_\gamma(m, \Delta m) = d\hat{\rho}_\gamma(m, J_m.\Delta m) \text{ for all } m \in \mathtt{M}, \gamma \in \mathsf{Diff}^+\mathbb{S}^1$$

is equivalent to

$$J_{\hat{\rho}_\gamma(m)}.\hat{\omega}_\gamma.\Delta m = \hat{\omega}_\gamma.J_m.\Delta m \text{ for all } m \in \mathtt{M}, \gamma \in \mathsf{Diff}^+\mathbb{S}^1, \text{ and } \Delta m \in \mathtt{K}.$$

Substituting $m = \hat{\Pi}(\gamma_1)$ and using

$$\hat{\rho}_\gamma(\hat{\Pi}(\gamma_1)) \overset{8.1.9}{=} \hat{\omega}_\gamma.\hat{\Pi}(\gamma_1) + \hat{\Pi}(\gamma^{-1}) \overset{8.1.8(ii)}{=} \hat{\Pi}(\gamma_1 \circ \gamma^{-1})$$

yields

$$J_{\hat{\Pi}(\gamma_1 \circ \gamma^{-1})} \circ \hat{\omega}_\gamma = \hat{\omega}_\gamma \circ J_{\hat{\Pi}(\gamma_1)} \text{ for all } \gamma, \gamma_1 \in \mathsf{Diff}^+\mathbb{S}^1.$$

The substitution $\gamma = \gamma^{-1}$ results in

$$J_{\hat{\Pi}(\gamma_1 \circ \gamma)} = \hat{\omega}_{\gamma^{-1}} \circ J_{\hat{\Pi}(\gamma_1)} \circ \hat{\omega}_\gamma \text{ for all } \gamma, \gamma_1 \in \mathsf{Diff}^+\mathbb{S}^1.$$

(iii)$\Longrightarrow$(ii): Substituting $\gamma_1 = \mathsf{id}_{\mathbb{S}^1}$ into (iii) gives us

$$J_{\hat{\Pi}(\gamma)} = \hat{\omega}_{\gamma^{-1}} \circ J_0 \circ \hat{\omega}_\gamma,$$

which is statement (ii).
(ii)$\Longrightarrow$(iii):
In the following calculation we use the statement (ii) in the first and in the last last equation:

$$\begin{aligned} J_{\hat{\Pi}(\gamma_1 \circ \gamma)} &= \hat{\omega}_{(\gamma_1 \circ \gamma)^{-1}} \circ J_0 \circ \hat{\omega}_{\gamma_1 \circ \gamma} \\ &= \hat{\omega}_{\gamma^{-1}} \circ \hat{\omega}_{\gamma_1^{-1}} \circ J_0 \circ \hat{\omega}_{\gamma_1} \circ \hat{\omega}_\gamma \\ &= \hat{\omega}_{\gamma^{-1}} \circ J_{\hat{\Pi}(\gamma_1)} \circ \hat{\omega}_\gamma. \end{aligned}$$

□

Lemma 8.2.11. *Every almost complex structure on* $\mathtt{M}$ *which is invariant under* $\mathsf{Diff}^+\mathbb{S}^1$ *is of the form of Definition 8.2.8*

Proof. Let $J : \mathtt{M} \times \mathtt{K} \to \mathtt{K}$ be an almost complex structure which is invariant under $\mathsf{Diff}^+\mathbb{S}^1$. By Lemma 8.2.10, we have

$$J_{\hat{\Pi}(\gamma)} = \hat{\omega}_{\gamma^{-1}} \circ J_0 \circ \hat{\omega}_\gamma$$

for all $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$, and

$$J_0 = \hat{\omega}_{\xi^{-1}} \circ J_0 \circ \hat{\omega}_\xi$$

for all $\xi \in \mathsf{Rot}^+\mathbb{S}^1$ due to $\hat{\Pi}(\xi) = 0$. Therefore, J_0 is of the form of Definition 8.2.7 due to Lemma 8.2.9. Hence, J is of the form of Definition 8.2.8. $\square$

Lemma 8.2.12. *If the almost complex structure as defined in 8.2.8 is integrable, then* $\sigma : \mathbb{N} \to \{-1, 1\}$ *is a constant function.*

Proof. 1. Step:
We make an indirect proof. Assume $\sigma : \mathbb{N} \to \{-1, 1\}$ is not the constant function and let

$$\tilde{\sigma} : \mathbb{Z}\backslash\{0\} \to \{-1, 1\}, \quad z \mapsto \left\{ \begin{array}{r|r} \sigma(z) & z \in \mathbb{N} \\ -\sigma(-z) & -z \in \mathbb{N} \end{array} \right. ,$$

then there exist two integers $k, l \in \mathbb{Z}\backslash\{0\}$ with $k \neq l$, $\sigma(k) = \sigma(l) = 1$ and $\sigma(k + l) = -1$. To find such k and l, just choose them with different sign.
2. Step:
We define

$$\hat{\text{II}} : \mathtt{M} \to \mathsf{Diff}^+\mathbb{S}^1, \quad m \mapsto \{t \mapsto (t + m(t))\}^{-1},$$

which is a section of

$$\hat{\Pi} : \mathsf{Diff}^+\mathbb{S}^1 \to \mathtt{M}, \quad \gamma \mapsto (1 - \hat{\mathtt{P}}_0)(\gamma - \mathsf{id}_{\mathbb{S}^1})$$

i.e.,

$$\hat{\Pi} \circ \hat{\text{II}} = \mathsf{id}_{\mathtt{M}}.$$

We can write the almost complex structure from Definition 8.2.8 as

$$(J_\sigma)_m = \hat{\omega}_{\hat{\Pi}(m)} \circ j_\sigma \circ \hat{\omega}_{\hat{\Pi}(m)^{-1}}.$$

Define $b_k \in \mathtt{K}_{\mathbb{C}}$ by $b_k(t) = e^{ikt}$. We deduce $j_\sigma.b_k = (i)^{\tilde{\sigma}(k)} b_k$ from Definition 8.2.7. Furthermore, for every $k \in \mathbb{Z}\backslash\{0\}$ we define a complex vector field on $\mathtt{M}$ by

$$\xi_k : \mathtt{M} \to \mathtt{K}_{\mathbb{C}}, \quad m \mapsto \hat{\omega}_{\hat{\Pi}(m)}.b_k.$$

This defines a vector field in the sense of

$$\xi_k : \mathtt{M} \to T(\mathtt{M}) \otimes \mathbb{C}$$

with the identifications

$$T(\mathtt{M}) \otimes \mathbb{C} = \bigcup_{m \in \mathtt{M}} T_m(\mathtt{M}) \otimes \mathbb{C}$$

and $T_m(\mathtt{M}) \otimes \mathbb{C} \simeq \mathtt{K}_{\mathbb{C}}$. Now we compute the action of the almost complex structure

$$J_\sigma : T(\mathtt{M}) \to T(\mathtt{M})$$

on the vector field $\xi_k : \mathtt{M} \to T(\mathtt{M}) \otimes \mathbb{C}$. For the point $m \in \mathtt{M}$ we have

$$J_m.\xi_k(m) = \hat{\omega}_{\hat{\Pi}}.j_\sigma.b_k = \hat{\omega}_{\hat{\Pi}}.(i)^{\tilde{\sigma}(k)} b_k = (i)^{\tilde{\sigma}(k)} \hat{\omega}_{\hat{\Pi}}.b_k = (i)^{\tilde{\sigma}(k)} \xi_k(m).$$

Hence, the fact $\tilde{\sigma}(n) = 1$ implies that ξ_n is a holomorphic vector field, i.e., $\xi_n \in \Gamma(T^+(\mathtt{M}))$. But, if $\tilde{\sigma}(n) = -1$, then ξ_n is an anti-holomorphic vector field, i.e., $\xi_n \in \Gamma(T^-(\mathtt{M}))$.
3. Step:
Let $k, l \in \mathbb{Z}\backslash\{0\}$ be the two different integers chosen in the first step. Since $\sigma(k) = \sigma(l) = 1$, the second step yields

$$\xi_k, \xi_l \in \Gamma(T^+(\mathtt{M})),$$

in words; ξ_k and ξ_l are holomorphic. In the rest of the proof we will show that their commutator

$$[\xi_k, \xi_l]$$

is not holomorphic anymore.

We have

$$\xi_k(m) = (1 - \hat{\mathtt{P}}_0)\{t \mapsto e^{ik(t+m(t))}\}$$

and the derivative is

$$d\xi_k(m, \Delta m) = (1 - \hat{\mathtt{P}}_0)\{t \mapsto ik\Delta m(t)e^{ik(t+m(t))}\}.$$

Moreover, we have

$$\begin{aligned} d\xi_k(m, \xi_l(m)) &= (1 - \hat{\mathtt{P}}_0)\left[\{t \mapsto ike^{ik(t+m(t))}\}(1 - \hat{\mathtt{P}}_0)\{t \mapsto e^{il(t+m(t))}\}\right] \\ &= (1 - \hat{\mathtt{P}}_0)\left[\{t \mapsto ike^{i(k+l)(t+m(t))}\}\right] \\ &\quad -(1 - \hat{\mathtt{P}}_0)\left[\{t \mapsto ike^{ik(t+m(t))}\}\right] \cdot \hat{\mathtt{P}}_0\{t \mapsto e^{il(t+m(t))}\}, \end{aligned}$$

which implies

$$\begin{aligned} [\xi_k, \xi_l](m) &= d\xi_k(m, \xi_l(m)) - d\xi_l(m, \xi_k(m)) \\ &= (1 - \hat{\mathtt{P}}_0)\left[\{t \mapsto i(k-l)e^{i(k+l)(t+m(t))}\}\right] \\ &\quad -(1 - \hat{\mathtt{P}}_0)\left[\{t \mapsto ike^{ik(t+m(t))}\}\right] \cdot \hat{\mathtt{P}}_0\{t \mapsto e^{il(t+m(t))}\} \\ &\quad +(1 - \hat{\mathtt{P}}_0)\left[\{t \mapsto ile^{il(t+m(t))}\}\right] \cdot \hat{\mathtt{P}}_0\{t \mapsto e^{ik(t+m(t))}\} \\ &= i(k-l)\xi_{k+l} \\ &\quad -ik\hat{\mathtt{P}}_0\{t \mapsto e^{il(t+m(t))}\} \cdot \xi_k + il\hat{\mathtt{P}}_0\{t \mapsto e^{ik(t+m(t))}\} \cdot \xi_l. \end{aligned}$$

Since $k - l \neq 0$ and $\tilde{\sigma}(k + l) = -1$ the first summand $i(k - l)\xi_{k+l}$ is a non-vanishing anti-holomorphic vector field, i.e., $i(k - l)\xi_{k+l} \in \Gamma(T^-(\mathtt{M}))$. The remaining term is a pure holomorphic vector field. Therefore, the commutator $[\xi_k, \xi_l]$ is not holomorphic. Hence, the almost complex structure J_σ with a non-constant σ is not integrable. □

Recall by Lemma 7.3.13 the Hilbert transform

$$\hat{\mathtt{H}} : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \lambda \mapsto \hat{\mathtt{H}}.\lambda$$

is

$$(\hat{\mathtt{H}}.\lambda)(\tau) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \frac{\lambda(\tau) - \lambda(\theta)}{\tan \frac{\tau - \theta}{2}} d\theta.$$

If we restrict $\hat{\mathtt{H}}$ to $\mathtt{K}$, then $\hat{\mathtt{H}}$ defines a complex structure on the vector space $\mathtt{K}$.

Definition 8.2.13 (Almost complex structure on $\mathtt{M}$)**.** We define the almost complex structure

$$\hat{\mathtt{J}} : \mathtt{M} \times \mathtt{K} \to \mathtt{K}, \quad (m, b) \mapsto \hat{\mathtt{J}}_m.b$$

by

$$\hat{\mathtt{J}}_{\hat{\Pi}(\gamma)}.b := -\hat{\omega}_{\gamma^{-1}}.\hat{\mathtt{H}}.\hat{\omega}_{\gamma}.b,$$

where we have chosen $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$ such that $\hat{\Pi}(\gamma) = m$.

Remark: $\hat{\mathtt{J}}$ is invariant under the action $\hat{\rho}$.

Proposition 8.2.14. *If there exists an almost complex structure on* $\mathtt{M}$ *which is invariant and integrable, then it is equal to* $\hat{\mathtt{J}}$ *or equal to* $-\hat{\mathtt{J}}$*.*

Proof. By Lemma 8.2.11, every almost complex structure which is $\mathsf{Diff}^+\mathbb{S}^1$-invariant is determined by a function $\sigma : \mathbb{N} \to \{-1, 1\}$ and is of the form

$$J_\sigma : \mathtt{M} \times \mathtt{K} \to \mathtt{K}, \quad (b, b) \mapsto (J_\sigma)_m.b$$

with

$$(J_\sigma)_m = \hat{\omega}_{\gamma^{-1}} \circ j_\sigma \circ \hat{\omega}_\gamma$$

chosen $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$ such that $\hat{\Pi}(\gamma) = m$, for a function $\sigma : \mathbb{N} \to \{-1, 1\}$. Due to Lemma 8.2.12 the function $\sigma : \mathbb{N} \to \{-1, 1\}$ is constant. Assume $\sigma \equiv 1$. By the Definition 8.2.7 and Lemma 7.3.7, the complex structure j_σ coincides with the Hilbert transform $\hat{\mathtt{H}}$. $\square$

Remark: So far, it is not shown that $\hat{\mathtt{J}}$ is integrable, because the existence of an invariant integrable almost-complex structure on $\mathtt{M}$ is not verified. The rest of this chapter is devoted to prove that $\hat{\mathtt{J}}$ is a complex structure, and the conclusion is possible that $\hat{\mathtt{J}}$ is integrable (compare with Lemma 8.2.6).

8.3 The Chart

To give $\mathtt{M}$ (and hence $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$) the structure of a complex manifold[1], we need a holomorphic atlas. Our holomorphic atlas consists of one chart

$$\hat{\Xi} : \mathtt{V}^- \to \mathtt{M}.$$

This chart will be constructed in this section (8.3.3), using the results and formulas of the preceding chapters. Furthermore, in this section, we will show that $\hat{\Xi}$ is bijective. The rest of this chapter is devoted to show that $\hat{\Xi}$ is a tame diffeomorphism and holomorphic.

Lemma 8.3.1. *For every tuple* $(\eta, \rho) \in \mathtt{V}^E \times \mathsf{Rot}^+\mathbb{S}^1$ *there exists exactly one tuple* $(c, f) \in \mathbb{C}^\times \times \mathtt{V}^+$ *such that*

$$\eta \circ \rho = c \cdot f.$$

Conversely, for every $(c, f) \in \mathbb{C}^\times \times \mathtt{V}^+$ *there exists exactly one tuple*

$$(\eta, \rho) \in \mathtt{V}^E \times \mathsf{Rot}^+\mathbb{S}^1$$

which satisfies the equation above.

Proof. Let $\eta \in \mathtt{V}^E$, then its Fourier series is

$$\eta(t) = e^{it}(r + a_1 e^{it} + a_2 e^{2it} + \cdots),$$

and a diffeomorphism $\rho \in \mathsf{Rot}^+\mathbb{S}^1$ has the form

$$\rho(t) = t + s$$

with $s \in \mathbb{S}^1$. On the other hand, for a function $f \in \mathtt{V}^+$ the Fourier series is

$$f(t) = e^{it}(1 + b_1 e^{it} + b_2 e^{2it} + \cdots).$$

If we substitute $c = r \cdot e^{is}$ and $b_k = \frac{a_k}{r} e^{iks}$ for $k = 1, 2, \ldots$, then

$$\begin{aligned} (\eta \circ \rho)(t) &= re^{is} \cdot e^{it}(r + \frac{a_1}{r} e^{is} \cdot e^{it} + \frac{a_2}{r} e^{2is} \cdot e^{2it} + \cdots) \\ &= c \cdot e^{it}(1 + b_1 e^{it} + b_2 e^{2it} + \cdots) \\ &= c \cdot f(t) \end{aligned}$$

proves the lemma. □

[1] It should be mentioned the article of Laszlo Lembert [18], where the Virasoro group Vir is considered as a complex manifold and the projection $\mathsf{Vir} \to \mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ as a holomorphic line bundle.

Lemma 8.3.2. *The function*

$$\phi : \mathtt{V}^- \to \mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$$

defined by

$$\phi(g) := \hat{\Gamma}(g)^{-1} \circ \mathsf{Rot}^+\mathbb{S}^1$$

is bijective.

Proof. 1. Step: Injectivity:
Let $\eta_1, \eta_2 \in \mathtt{V}^-$ with $\phi(\eta_1) = \phi(\eta_2)$. Then

$$\hat{\Gamma}(\eta_1)^{-1} \circ \mathsf{Rot}^+\mathbb{S}^1 = \hat{\Gamma}(\eta_2)^{-1} \circ \mathsf{Rot}^+\mathbb{S}^1$$

and there exists a $\rho \in \mathsf{Rot}^+\mathbb{S}^1$ such that

$$\hat{\Gamma}(\eta_1) = \rho \circ \hat{\Gamma}(\eta_2)$$

holds. By Corollary 4.2.3(i), we have

$$\eta_1 = \hat{\Lambda}(\eta_1) \circ \hat{\Gamma}(\eta_1) = \hat{\Lambda}(\eta_1) \circ \rho \circ \hat{\Gamma}(\eta_2).$$

Since $\hat{\Lambda}(\eta_1) \in \mathtt{V}^E$, Lemma 8.3.1 yields a complex number $c \in \mathbb{C}^\times$ and a function $f \in \mathtt{V}^+$ such that $\hat{\Lambda}(\eta_1) \circ \rho = c \cdot f$ holds. Hence,

$$\eta_1 = c \cdot f \circ \hat{\Gamma}(\eta_2). \qquad (A)$$

On the other hand we apply Corollary 4.2.3(i) to η_2 and get

$$\eta_2 = \hat{\Lambda}(\eta_2) \circ \hat{\Gamma}(\eta_2).$$

By Lemma 8.3.1, there exists a complex number $\tilde{c} \in \mathbb{C}^\times$ and a function $\tilde{f} \in \mathtt{V}^+$ such that

$$\hat{\Lambda}(\eta_2) \circ \mathsf{id}_{\mathbb{S}^1} = \tilde{c} \cdot \tilde{f}.$$

Hence,

$$\eta_2 = \tilde{c} \cdot \tilde{f} \circ \hat{\Gamma}(\eta_2). \qquad (B)$$

By Theorem 3.4.1, we obtain that (A) and (B) are the same Birkhoff decompositions of $\hat{\Gamma}(\eta_2)$, and we conclude $\eta_1 = \eta_2$, which proves that ϕ is injective.

2. Step: surjectivity:
Let $[\gamma] \in \mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ be the equivalence class of $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$. By Theorem 3.4.1, there exist two functions $f \in \mathtt{V}^+$, $g \in \mathtt{V}^-$ and a complex number $c \in \mathbb{C}^\times$ such that

$$g = c \cdot f \circ \gamma^{-1}.$$

It remains to show that $\phi(g) = [\gamma]$. By Lemma 8.3.1, there exists a function $\eta \in \mathtt{V}^E$ and a diffeomorphism $\xi \in \mathsf{Rot}^+\mathbb{S}^1$ with $c \cdot f = \eta \circ \xi$. Therefore,

$$g = \eta \circ (\xi \circ \gamma^{-1}).$$

The bijectivity of the composition map (see 4.2.1)

$$\mathtt{C} : \mathtt{V}^E \times \mathsf{Diff}^+\mathbb{S}^1 \to \mathtt{V}, \quad (\eta, \gamma) \mapsto \eta \circ \gamma$$

implies $\hat{\Gamma}(g) = \xi \circ \gamma^{-1}$ and $\hat{\Gamma}(g)^{-1} \circ \xi = \gamma$. We conclude

$$\phi(g) = [\hat{\Gamma}(g)^{-1}] = [\gamma].$$

$\square$

Definition 8.3.3. We define the function

$$\hat{\Xi} : \mathtt{V}^- \to \mathtt{M}, \quad \eta \mapsto \hat{\Xi}(\eta) := \hat{\Pi}(\hat{\Gamma}(\eta)).$$

Remark: $\hat{\Pi}$ was defined in 8.1.3 by

$$\hat{\Pi}(\gamma) = (1 - \hat{\mathtt{P}}_0).(\gamma - \mathsf{id}_{\mathbb{S}^1}) = (\gamma - \mathsf{id}_{\mathbb{S}^1}) - \int_{\mathbb{S}^1} (\gamma(t) - t)dt,$$

and $\hat{\Gamma}$ was defined in 4.2.2 to be the projection to the right side of the inverse of the composition map.

Lemma 8.3.4. *The function* $\hat{\Xi} : \mathtt{V}^- \to \mathtt{M}$ *is bijective.*

Proof. By Lemma 8.1.6, the function

$$\hat{\Psi} : \mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1 \to \mathtt{M}, \quad [\gamma] \mapsto \hat{\Pi}(\gamma^{-1})$$

is bijective and by Lemma 8.3.2 the function

$$\phi : \mathtt{V}^- \to \mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1, \quad g \mapsto \hat{\Gamma}(g)^{-1} \circ \mathsf{Rot}^+\mathbb{S}^1$$

is bijective. The function $\hat{\Xi}$ is the composition

$$\hat{\Xi} = \hat{\Psi} \circ \phi$$

of these two bijective function, and hence it is bijective itself. $\square$

Lemma 8.3.5 (Properties of $\hat{\Upsilon}$). *The function*

$$\hat{\Upsilon} : \mathtt{V} \to \mathtt{V}, \quad f \mapsto \hat{\Upsilon}(f)$$

with $\hat{\Upsilon}(f)(t) = 1/f(-t)$ *for all* $t \in \mathbb{S}^1$ *has the following properties:*

(i) *It is smooth-tame.*

(ii) *Its derivative is*

$$d\hat{\Upsilon} : V \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad (f, \Delta f) \to d\hat{\Upsilon}(f, \Delta f)$$

with $d\hat{\Upsilon}(f, \Delta f)(t) = -(\Delta f(-t))/(f(-t)^2)$ *for* $t \in \mathbb{S}^1$.

(iii) *It is holomorphic.*

Proof. Define two smooth functions

$$\begin{aligned} a : \mathbb{C}^\times \to \mathbb{C}^\times, &\qquad z \mapsto \frac{1}{z}, \qquad \text{and} \\ b : \mathbb{S}^1 \to \mathbb{S}^1, &\qquad t \mapsto -t. \end{aligned}$$

Since $\hat{\Upsilon}(f) = a \circ \hat{\Upsilon}(f) \circ b$, we can write $\hat{\Upsilon} = \mathsf{L}_a \circ \mathsf{R}_b$.
(i): The left composition

$$\mathsf{L}_a : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times), \quad f \mapsto a \circ f$$

is smooth-tame by Lemma 5.3.5, and the right composition

$$\mathsf{R}_b : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad f \mapsto f \circ b$$

is smooth-tame by Lemma 5.2.11. We conclude by Lemma 5.2.8 that the composition $\hat{\Upsilon} = \mathsf{L}_a \circ \mathsf{R}_b$ is smooth-tame.
(ii): The derivative of a is

$$da(z, \Delta z) = -\frac{\Delta z}{z^2},$$

and applying the chain rule to $\hat{\Upsilon} = \mathsf{L}_a \circ \mathsf{R}_b$ yields

$$\begin{aligned} d\hat{\Upsilon}(f, \Delta f)(t) &= d\mathsf{L}_a(\mathsf{R}_b(f), d\mathsf{R}_b(f, \Delta f))(t) \\ &\overset{5.5.1(ii)}{=} da(\mathsf{R}_b(f)(t), d\mathsf{R}_b(f, \Delta(f))(t)) \\ &\overset{5.5.1(i)}{=} da(f(b(t)), \Delta f(b(t))\,) \\ &= -\frac{\Delta f(b(t))}{f(b(t))^2} \\ &= -\frac{\Delta f(-t)}{f(-t)^2}. \end{aligned}$$

(iii) Using the formula for the derivative of $\hat{\Upsilon}$ in part (i) of this lemma we obtain

$$d\hat{\Upsilon}(f, i \cdot \Delta f) = i \cdot d\hat{\Upsilon}(f, \Delta f).$$

$\square$

With the preceding lemma we can also define an alternative chart

$$\hat{\Xi} \circ \hat{\Upsilon} : \mathtt{V}^+ \to \mathtt{M}$$

taking into account that the restriction $\hat{\Upsilon} : \mathtt{V}^+ \to \mathtt{V}^-$ is bijective by Lemma 2.4.10.

8.4 Derivative of $\hat{\Xi}$

The purpose of this section is to compute the derivative of $\hat{\Xi}$. The result of this section (Proposition 8.4.14) is that the derivative of $\hat{\Xi}$ can be written as a composition of three functions. This is one item in the proof of the fact that $\hat{\Xi} : \mathtt{V}^- \to \mathtt{M}$ is a tame diffeomorphism.

Definition 8.4.1. Let $f \in \mathtt{V}$ and $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$. Then, we define

$$[\hat{\mathtt{h}}_f.\lambda](\tau) := \frac{1}{\hat{\Lambda}(f)'(\tau)} \cdot \lambda(\hat{\Gamma}(f)^{-1}(\tau)).$$

We recollect that by Definition 2.2.3(ii) the subset $\mathtt{F}^- \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is the subspace of functions of the form

$$f(t) = a_0 + a_1 e^{-it} + \cdots$$

with $a_1, a_2, \ldots \in \mathbb{C}$. Moreover, we define the subspace

$$\mathtt{H}_f := \{\, \hat{\mathtt{h}}_f.\lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \;:\; \lambda \in \mathtt{F}^- \,\}.$$

Remark: From the definition above we obtain that

$$\hat{\mathtt{h}}_f : \mathtt{F}^- \to \mathtt{H}_f, \quad \lambda \mapsto \hat{\mathtt{h}}_f.\lambda$$

is a bijective function, whose inverse is

$$\hat{\mathtt{h}}_f^{-1} : \mathtt{H}_f \to \mathtt{F}^-, \quad \lambda \mapsto \hat{\mathtt{h}}_f^{-1}(\lambda) = [\hat{\Lambda}(f)' \circ \hat{\Gamma}(f)] \cdot [\lambda \circ \hat{\Gamma}(f)].$$

Definition 8.4.2. For $f \in \mathtt{V}$, we define the function

$$\zeta_f : \mathbb{S}^1 \to \mathbb{C}, \quad t \mapsto -i\hat{\Lambda}(f)'(t) \cdot e^{-it}.$$

Lemma 8.4.3. *If $f \in \mathtt{V}$, then $\hat{l}(\zeta_f) = \hat{\mathtt{P}}_F.\hat{l}(\zeta_f)$.*

Proof. 1. Step: $\zeta_f \in \mathtt{W}$:
Since $\hat{\Lambda}(f) \in \mathtt{V}$, we have $\hat{\Lambda}(f)'(t) \neq 0$ for all $t \in \mathbb{S}^1$, such that $\zeta_f(t) \neq 0$ for all $t \in \mathbb{S}^1$. Moreover, the winding number is

$$\mathtt{w}(\zeta_f) = \mathtt{w}(\hat{\Lambda}(f)') + \mathtt{w}(\{t \mapsto e^{-it}\}) \overset{6.3.4}{=} 1 - 1 = 0.$$

So, the conditions of Definition 7.2.3 are fulfilled, such that $\zeta_f \in \mathtt{W}$.
2. Step: $\zeta_f \in \mathtt{F}^+$:
The Fourier series of $\hat{\Lambda}(f) \in \mathtt{V}^E$ is

$$\hat{\Lambda}(f)(t) = e^{it}(r + a_1 e^{it} + a_2 e^{2it} + \cdots)$$

with $r \in \mathbb{R}$ and $a_1, a_2, \ldots \in \mathbb{C}$, which implies

$$\zeta_f(t) = ir + 2ia_1 e^{it} + 3ia_2 e^{2it} + \cdots$$

and by Definition 2.2.3 we get $\zeta_f \in \mathtt{F}^+$.
3. Step:
So far, we have shown that $\in \mathtt{W} \cap \mathtt{F}^+$ and Lemma 7.4.8 yields $\hat{l}(\zeta_f) = \hat{\mathtt{P}}_F.\hat{l}(\zeta_f)$.
□

Lemma 8.4.4. *If $f \in \mathtt{V}$, then*

$$e^{2[\hat{\mathtt{P}}_F.\hat{\mathtt{P}}_I.\hat{l}(\hat{k}_f(\hat{\Lambda}(f)))](\tau)} = \frac{\hat{\mathtt{m}}(|\hat{\Lambda}(f)'|)}{\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f)))} \cdot \frac{i}{\hat{\Lambda}(f)'(\tau)} e^{i\tau}$$

for all $\tau \in \mathbb{S}^1$.

Proof.

$$\begin{aligned}
& e^{2[\hat{\mathtt{P}}_F.\hat{\mathtt{P}}_I.\hat{l}(\hat{k}_f(\hat{\Lambda}(f)))](\tau)} \overset{7.2.8(ii)}{=} e^{2[\hat{\mathtt{P}}_F \hat{\mathtt{P}}_I \hat{l}(\frac{|\zeta_f|}{\zeta_f})](\tau)} \\
\overset{7.4.7(i)+(ii)}{=}\ & e^{2[\hat{\mathtt{P}}_F \hat{\mathtt{P}}_I (\hat{l}(|\zeta_f|) - \hat{l}(\zeta_f))](\tau)} \overset{7.4.1}{=} e^{2[\hat{\mathtt{P}}_F \hat{\mathtt{P}}_I (-\hat{l}(\zeta_f))](\tau)} \\
\overset{8.4.3}{=}\ & e^{-2[\hat{\mathtt{P}}_F \hat{\mathtt{P}}_I \hat{\mathtt{P}}_F.\hat{l}(\zeta_f)](\tau)} \overset{7.3.3(i)}{=} e^{-[\hat{\mathtt{P}}_F.\hat{l}(\zeta_f)](\tau)} \\
\overset{8.4.3}{=}\ & e^{-[\hat{l}(\zeta_f)](\tau)} \\
\overset{7.4.2}{=}\ & \hat{\mathtt{m}}(\zeta_f) \cdot \frac{1}{\zeta_f(\tau)} \\
\overset{7.2.8(ii)}{=}\ & \hat{\mathtt{m}}\left(\frac{|\hat{\Lambda}(f)'|}{\hat{k}_f(\hat{\Lambda}(f))}\right) \cdot \frac{1}{\zeta_f(\tau)} \\
\overset{7.4.10(i)+(ii)}{=}\ & \frac{\hat{\mathtt{m}}(|\hat{\Lambda}(f)'|)}{\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f)))} \cdot \frac{1}{\zeta_f(\tau)} \\
\overset{8.4.2}{=}\ & \frac{\hat{\mathtt{m}}(|\hat{\Lambda}(f)'|)}{\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f)))} \cdot \frac{i}{\hat{\Lambda}(f)'(\tau)} e^{i\tau}.
\end{aligned}$$

□

Lemma 8.4.5. *If $f \in \mathtt{V}$, then*

$$\hat{\delta}^{\#}_f(\hat{\Lambda}(f),\lambda)(\tau) = -i\hat{\Lambda}(f)'(\tau)\cdot\frac{\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f)))}{\hat{\mathtt{m}}(|\hat{\Lambda}(f)'|)}\cdot\lambda(\tau)$$

for all $\lambda \in \mathtt{E}_0$ and $\tau \in \mathbb{S}^1$.

Proof.

$$\begin{aligned}\hat{\delta}^{\#}_f(\hat{\Lambda}(f),\lambda)(\tau) &\overset{7.5.4}{=} e^{-2[\hat{\mathtt{P}}_F.\hat{\mathtt{P}}_I.\hat{l}(\hat{k}_f(\hat{\Lambda}(f)))](\tau)}\cdot e^{i\tau}\lambda(\tau)\\ &\overset{8.4.4}{=} \frac{\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f)))}{\hat{\mathtt{m}}(|\hat{\Lambda}(f)'|)}\frac{\hat{\Lambda}(f)'(\tau)}{i}\frac{1}{e^{i\tau}}\cdot e^{i\tau}\lambda(\tau)\\ &= -i\hat{\Lambda}(f)'(\tau)\cdot\frac{\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f)))}{\hat{\mathtt{m}}(|\hat{\Lambda}(f)'|)}\cdot\lambda(\tau).\end{aligned}$$

□

Lemma 8.4.6. *If $f \in \mathtt{V}$, then*

$$\hat{\vartheta}^{\#}_f(\hat{\Lambda}(f),\lambda) = \frac{1}{\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f)))}\cdot[(2\hat{\mathtt{P}}_F+\hat{\mathtt{P}}_0).\lambda]$$

for $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{R})$.

Proof. We have

$$\begin{aligned}&\hat{\vartheta}^{\#}_f(\hat{\Lambda}(f),\lambda)(\tau)\\ \overset{7.5.6}{=}\ &\frac{1}{\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f)))}\cdot 2(\hat{\mathtt{P}}_F.\lambda)(\tau)+\frac{1}{\mathsf{Re}\,\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f)))}\cdot(\hat{\mathtt{P}}_0.\lambda)(\tau)\\ \overset{7.4.14}{=}\ &\frac{1}{\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f)))}\cdot 2(\hat{\mathtt{P}}_F.\lambda)(\tau)+\frac{1}{\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f)))}\cdot(\hat{\mathtt{P}}_0.\lambda)(\tau)\\ =\ &\frac{1}{\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f)))}\cdot[(2\hat{\mathtt{P}}_F+\hat{\mathtt{P}}_0).\lambda](\tau)\end{aligned}$$

for all $\tau \in \mathbb{S}^1$. □

Lemma 8.4.7. *If $f \in \mathtt{V}$, then*

$$\hat{\sigma}^{\#}_f(\hat{\Lambda}(f),\lambda)(\tau) = \frac{\hat{\mathtt{m}}(|\hat{\Lambda}(f)'|)}{|\hat{\Lambda}(f)'(\tau)|}\cdot\lambda(\hat{\Gamma}(f)^{-1}(\tau))$$

for $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{R})$ and $\tau \in \mathbb{S}^1$.

Proof.

$$
\begin{aligned}
&\hat{\sigma}_f^{\#}(\hat{\Lambda}(f),\lambda)(\tau)\\
\overset{7.5.7}{=}\ & e^{[-\hat{\mathtt{P}}_K \hat{l}(\hat{k}_f(\hat{\Lambda}(f)))](\tau)} \cdot \lambda([\hat{\Sigma}_f(\hat{\Lambda}(f))](\tau))\\
\overset{7.1.8(ii)}{=}\ & e^{[-\hat{\mathtt{P}}_K \hat{l}(\hat{k}_f(\hat{\Lambda}(f)))](\tau)} \cdot \lambda(\hat{\Gamma}(f)^{-1}(\tau))\\
\overset{7.3.3(ix)}{=}\ & e^{2[\hat{\mathtt{P}}_F.\hat{\mathtt{P}}_I.\hat{l}(\hat{k}_f(\hat{\Lambda}(f)))](\tau)} \cdot e^{-[(1-\hat{\mathtt{P}}_0).\hat{l}(\hat{k}_f(\hat{\Lambda}(f)))](\tau)} \cdot \lambda(\hat{\Gamma}(f)^{-1}(\tau))\\
\overset{7.4.5}{=}\ & e^{2[\hat{\mathtt{P}}_F.\hat{\mathtt{P}}_I.\hat{l}(\hat{k}_f(\hat{\Lambda}(f)))](\tau)} \cdot e^{-[\hat{l}(\hat{k}_f(\hat{\Lambda}(f)))](\tau)} \cdot \lambda(\hat{\Gamma}(f)^{-1}(\tau))\\
\overset{7.4.2}{=}\ & e^{2[\hat{\mathtt{P}}_F.\hat{\mathtt{P}}_I.\hat{l}(\hat{k}_f(\hat{\Lambda}(f)))](\tau)} \cdot \frac{\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f)))}{\hat{k}_f(\hat{\Lambda}(f))(\tau)} \cdot \lambda(\hat{\Gamma}(f)^{-1}(\tau))\\
\overset{8.4.4}{=}\ & \frac{\hat{\mathtt{m}}(|\hat{\Lambda}(f)'|)}{\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f)))} \cdot \frac{i}{\hat{\Lambda}(f)'(\tau)} e^{i\tau} \cdot \frac{\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f)))}{\hat{k}_f(\hat{\Lambda}(f))(\tau)} \cdot \lambda(\hat{\Gamma}(f)^{-1}(\tau))\\
\overset{7.2.8(ii)}{=}\ & \frac{\hat{\mathtt{m}}(|\hat{\Lambda}(f)'|)}{|\hat{\Lambda}(f)'(\tau)|} \cdot \lambda(\hat{\Gamma}(f)^{-1}(\tau)).
\end{aligned}
$$

□

Lemma 8.4.8. *If $f \in \mathtt{V}$, then*

$$
d\hat{\Theta}_f(f,\Delta\eta)(t) = \mathsf{Re}[i\frac{|f'(t)|}{f'(t)} \cdot \Delta\eta(t)].
$$

for all $\Delta\eta \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C})$ and all $t \in \mathbb{S}^1$.

Proof.

$$
\begin{aligned}
&d\hat{\Theta}_f(f,\Delta\eta)(t)\\
\overset{7.2.6}{=}\ & \mathsf{Re}[\hat{k}_f(f)(\tau) \cdot e^{-i\tau}\Delta\eta(\tau)]\big|_{\tau=[\hat{\Sigma}_f(f)]^{-1}(t)}\\
\overset{7.1.8(i)}{=}\ & \mathsf{Re}[\hat{k}_f(f)(t) \cdot e^{-it}\Delta\eta(t)]\\
\overset{7.2.8(i)}{=}\ & \mathsf{Re}[i\frac{|f'(t)|}{f'(t)} e^{it} \cdot e^{-it}\Delta\eta(t)]\\
=\ & \mathsf{Re}[i\frac{|f'(t)|}{f'(t)}\Delta\eta(t)].
\end{aligned}
$$

□

Lemma 8.4.9. *If $f \in \mathtt{V}$, then*

$$
\hat{\sigma}_f^{\#}(\hat{\Lambda}(f), d\hat{\Theta}_f(f,\Delta\eta)) = \hat{\mathtt{m}}(|\hat{\Lambda}(f)'|) \cdot [\hat{\mathtt{I}}.\hat{\mathtt{P}}_I.\hat{\mathtt{h}}_f.\Delta\eta]
$$

for all $\Delta\eta \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C})$.

Proof. We have

$$
\begin{aligned}
& \hat{\sigma}_f^{\#}(\hat{\Lambda}(f), d\hat{\Theta}_f(f,\Delta\eta))(\tau) \\
\overset{8.4.7}{=}\ & \frac{\hat{\mathtt{m}}(|\hat{\Lambda}(f)'|)}{|\hat{\Lambda}(f)'(\tau)|} \cdot d\hat{\Theta}_f(f,\Delta\eta)(\hat{\Gamma}(f)^{-1}(\tau)) \\
\overset{8.4.8}{=}\ & \hat{\mathtt{m}}(|\hat{\Lambda}(f)'|) \cdot \mathsf{Re}[i\frac{1}{|\hat{\Lambda}(f)'(\tau)|} \cdot \frac{|f'(\hat{\Gamma}(f)^{-1}(\tau))|}{f'(\hat{\Gamma}(f)^{-1}(\tau))} \cdot \Delta\eta(\hat{\Gamma}(f)^{-1}(\tau))] \\
\overset{4.2.3(ii)}{=}\ & \hat{\mathtt{m}}(|\hat{\Lambda}(f)'|) \cdot \mathsf{Re}[i\frac{1}{f'(\hat{\Gamma}(f)^{-1}(\tau))}\frac{1}{[\hat{\Gamma}(f)^{-1}]'(\tau)} \cdot \Delta\eta(\hat{\Gamma}(f)^{-1}(\tau))] \\
\overset{4.2.3(ii)}{=}\ & \hat{\mathtt{m}}(|\hat{\Lambda}(f)'|) \cdot \mathsf{Re}[i\frac{1}{\hat{\Lambda}(f)'(\tau)} \cdot \Delta\eta(\hat{\Gamma}(f)^{-1}(\tau))] \\
\overset{8.4.1}{=}\ & \hat{\mathtt{m}}(|\hat{\Lambda}(f)'|) \cdot \mathsf{Re}[i \cdot (\hat{\mathtt{h}}_f.\Delta\eta)(\tau))] \\
\overset{7.3.6}{=}\ & \hat{\mathtt{m}}(|\hat{\Lambda}(f)'|) \cdot [\hat{\mathtt{I}}.\hat{\mathtt{P}}_I.\hat{\mathtt{h}}_f.\Delta\eta](\tau).
\end{aligned}
$$

for all $\tau \in \mathbb{S}^1$. □

Proposition 8.4.10 (Derivative of $\hat{\Lambda}$). *The derivative of $\hat{\Lambda} : \mathtt{V} \to \mathtt{V}^E$ at the point $f \in \mathtt{V}$ is*

$$d\hat{\Lambda} : \mathtt{V} \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathbb{S}^1$$

with

$$d\hat{\Lambda}(f,\Delta\eta) = \hat{\Lambda}(f)' \cdot [(2\hat{\mathtt{P}}_F + \hat{\mathtt{P}}_0).\hat{\mathtt{P}}_I.\hat{\mathtt{h}}_f.\Delta\eta].$$

Proof.

$$
\begin{aligned}
& d\hat{\Lambda}(f,\Delta\eta) \\
\overset{7.1.10(i)}{=}\ & d(\hat{\theta}_f^{-1} \circ \hat{\Theta}_f)(f,\Delta\eta) \\
=\ & d[\hat{\theta}_f^{-1}](\hat{\Theta}_f(f), d\hat{\Theta}_f(f,\Delta\eta)\,) \\
=\ & [[d\hat{\theta}_f]_{\hat{\theta}_f^{-1}(\hat{\Theta}_f(f))}]^{-1}.d\hat{\Theta}_f(f,\Delta\eta) \\
\overset{7.1.10(i)}{=}\ & [[d\hat{\theta}_f]_{\hat{\Lambda}(f)}]^{-1}.d\hat{\Theta}_f(f,\Delta\eta) \\
\overset{7.5.8}{=}\ & \hat{\delta}_f^{\#}(\hat{\Lambda}(f), \hat{\vartheta}_f^{\#}(\hat{\Lambda}(f),\ \hat{\sigma}_f^{\#}(\hat{\Lambda}(f), d\hat{\Theta}_f(f,\Delta\eta)\,))) \\
\overset{8.4.9}{=}\ & \hat{\delta}_f^{\#}(\hat{\Lambda}(f), \hat{\vartheta}_f^{\#}(\hat{\Lambda}(f),\ \hat{\mathtt{m}}(|\hat{\Lambda}(f)'|)[\hat{\mathtt{I}}.\hat{\mathtt{P}}_I.\hat{\mathtt{h}}_f.\Delta\eta]\,)) \\
\overset{8.4.6}{=}\ & \hat{\delta}_f^{\#}(\hat{\Lambda}(f), \frac{\hat{\mathtt{m}}(|\hat{\Lambda}(f)'|)}{\hat{\mathtt{m}}(\hat{k}_f(\hat{\Lambda}(f)))}[(2\hat{\mathtt{P}}_F + \hat{\mathtt{P}}_0).\hat{\mathtt{I}}.\hat{\mathtt{P}}_I.\hat{\mathtt{h}}_f.\Delta\eta]) \\
\overset{8.4.5}{=}\ & -i\hat{\Lambda}(f)' \cdot [(2\hat{\mathtt{P}}_F + \hat{\mathtt{P}}_0).\hat{\mathtt{I}}.\hat{\mathtt{P}}_I.\hat{\mathtt{h}}_f.\Delta\eta] \\
\overset{7.3.2}{=}\ & \hat{\Lambda}(f)' \cdot [(2\hat{\mathtt{P}}_F + \hat{\mathtt{P}}_0).\hat{\mathtt{P}}_I.\hat{\mathtt{h}}_f.\Delta\eta].
\end{aligned}
$$

In the last step, we have used that the projections $\hat{\mathtt{P}}_F$ and $\hat{\mathtt{P}}_0$ are complex linear. □

Lemma 8.4.11. *Let $f \in \mathtt{V}$. If $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$ and $\Delta\eta \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$, then*

$$d\hat{\Sigma}_f(f \circ \gamma^{-1}, \Delta\eta)(\gamma(t)) = \mathsf{Re}(\frac{\Delta\eta(\gamma(t))}{f'(t)})$$

for all $t \in \mathbb{S}^1$.

Proof.

$$\begin{aligned}
& d\hat{\Sigma}_f(f \circ \gamma^{-1}, \Delta\eta)(\gamma(t)) \\
\overset{7.1.3}{=}\ & [d\mathsf{L}_{\pi_2 \circ \mathtt{N}_f^{-1}}(f \circ \gamma^{-1}, \Delta\eta)](\gamma(t)) \\
\overset{5.5.1(i)}{=}\ & d[\pi_2 \circ \mathtt{N}_f^{-1}](f(t), \Delta\eta(\gamma(t))) \\
\overset{6.2.4(ii)}{=}\ & \mathsf{Re}(\frac{|f'(\pi_2(\mathtt{N}_f^{-1}(f(t))))|}{f'(\pi_2(\mathtt{N}_f^{-1}(f(t))))} \cdot \frac{\Delta\eta(\gamma(t))}{\det d\mathtt{N}_f(\pi_1(\mathtt{N}_f^{-1}(f(t))), \pi_2(\mathtt{N}_f^{-1}(f(t))))}) \\
\overset{6.1.10(ii)}{=}\ & \mathsf{Re}(\frac{|f'(t)|}{f'(t)} \cdot \frac{\Delta\eta(\gamma(t))}{\det d\mathtt{N}_f(\pi_1(\mathtt{N}_f^{-1}(f(t))), t)}) \\
\overset{6.1.10(i)}{=}\ & \mathsf{Re}(\frac{|f'(t)|}{f'(t)} \cdot \frac{\Delta\eta(\gamma(t))}{\det d\mathtt{N}_f(0, t)}) \\
\overset{6.2.2(i)}{=}\ & \mathsf{Re}(\frac{|f'(t)|}{f'(t)} \cdot \frac{\Delta\eta(\gamma(t))}{|f'(t)|}) \\
=\ & \mathsf{Re}(\frac{1}{f'(t)} \cdot \Delta\eta(\gamma(t))).
\end{aligned}$$

□

Proposition 8.4.12. *The derivative of $\hat{\Gamma} : \mathtt{V} \to \mathsf{Diff}^+\mathbb{S}^1$ is*

$$d\hat{\Gamma} : \mathtt{V} \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$$

with

$$d\hat{\Gamma}(f, \Delta\eta) = \mathsf{R}_{\hat{\Gamma}(f)}.\hat{\mathtt{P}}_R.[1 - (2\hat{\mathtt{P}}_F + \hat{\mathtt{P}}_0).\hat{\mathtt{P}}_I].\hat{\mathtt{h}}_f.\Delta\eta$$

for all $f \in \mathtt{V}$ and $\Delta\eta \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$.

Proof. Lemma 7.1.10 yields

$$\begin{aligned}
\hat{\Gamma}(\eta) &= [(\hat{\Sigma}_f \circ \hat{\Lambda})(\eta)]^{-1} \circ \hat{\Sigma}_f(\eta) \\
&= \iota(\hat{\Sigma}_f(\hat{\Lambda}(\eta))) \circ \hat{\Sigma}_f(\eta),
\end{aligned}$$

and we compute

$$
\begin{aligned}
& d\hat{\Gamma}(f,\Delta\eta)(t) \\
\overset{5.5.4}{=}\ & d[\hat{\Sigma}_f(\hat{\Lambda}(f))^{-1}]_{\hat{\Sigma}_f(f)(t)} \times \\
& \qquad \left\{ d\hat{\Sigma}_f(f,\Delta\eta)(t) - d\hat{\Sigma}_f(\hat{\Lambda}(f), d\hat{\Lambda}(f,\Delta\eta)\,)(\hat{\Gamma}(f)(t)) \right\} \\
\overset{7.1.8(ii)}{=}\ & d[\hat{\Gamma}(f)]_{\hat{\Sigma}_f(f)(t)} \times \\
& \qquad \left\{ d\hat{\Sigma}_f(f,\Delta\eta)(t) - d\hat{\Sigma}_f(\hat{\Lambda}(f), d\hat{\Lambda}(f,\Delta\eta)\,)(\hat{\Gamma}(f)(t)) \right\} \\
\overset{7.1.8(i)}{=}\ & d[\hat{\Gamma}(f)]_t \left\{ d\hat{\Sigma}_f(f,\Delta\eta)(t) - d\hat{\Sigma}_f(\hat{\Lambda}(f), d\hat{\Lambda}(f,\Delta\eta)\,)(\hat{\Gamma}(f)(t)) \right\} \\
\overset{4.2.3(i)}{=}\ & \hat{\Gamma}(f)'(t) \left\{ d\hat{\Sigma}_f(f,\Delta\eta)(t) - d\hat{\Sigma}_f(f\circ\hat{\Gamma}(f)^{-1}, d\hat{\Lambda}(f,\Delta\eta)\,)(\hat{\Gamma}(f)(t)) \right\} \\
\overset{8.4.11}{=}\ & \hat{\Gamma}(f)'(t)\mathsf{Re}\left(\frac{1}{f'(t)} \left\{ \Delta\eta(t) - d\hat{\Lambda}(f,\Delta\eta)(\hat{\Gamma}(f)(t)) \right\} \right) \\
\overset{4.2.3(ii)}{=}\ & \mathsf{Re}\left(\frac{1}{\hat{\Lambda}(f)'(\hat{\Gamma}(f)(t))} \left\{ \Delta\eta(t) - d\hat{\Lambda}(f,\Delta\eta)(\hat{\Gamma}(f)(t)) \right\} \right) \\
\overset{8.4.1}{=}\ & \mathsf{Re}\left((\hat{\mathtt{h}}_f.\Delta\eta)(\hat{\Gamma}(f)(t)) - \frac{1}{\hat{\Lambda}(f)'(\hat{\Gamma}(f)(t))} \left\{ d\hat{\Lambda}(f,\Delta\eta)(\hat{\Gamma}(f)(t)) \right\} \right) \\
\overset{8.4.10}{=}\ & \mathsf{Re}\left((\hat{\mathtt{h}}_f.\Delta\eta)(\hat{\Gamma}(f)(t)) - [(2\hat{\mathtt{P}}_F + \hat{\mathtt{P}}_0).\hat{\mathtt{P}}_I.\hat{\mathtt{h}}_f.\Delta\eta](\hat{\Gamma}(f)(t)) \right) \\
=\ & \left\{ \mathsf{R}_{\hat{\Gamma}(f)}.\hat{\mathtt{P}}_R.[1-(2\hat{\mathtt{P}}_F + \hat{\mathtt{P}}_0).\hat{\mathtt{P}}_I].\hat{\mathtt{h}}_f.\Delta\eta \right\}(t)
\end{aligned}
$$

for all $t \in \mathbb{S}^1$. $\square$

Lemma 8.4.13. *If $f \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C})$ and $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$, then*

$$(1-\hat{\mathtt{P}}_0).\mathsf{R}_\gamma.f = (1-\hat{\mathtt{P}}_0).\mathsf{R}_\gamma.(1-\hat{\mathtt{P}}_0).f$$

where $\mathsf{R}_\gamma.f$ is an abbreviation for $f\circ\gamma$.

Proof. The function $\hat{\mathtt{P}}_0.f : \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C})$ is the constant function and therefore $\mathsf{R}_\gamma.\hat{\mathtt{P}}_0.f = \hat{\mathtt{P}}_0.f$. The calculation

$$
\begin{aligned}
(1-\hat{\mathtt{P}}_0).\mathsf{R}_\gamma.(1-\hat{\mathtt{P}}_0).f =& (1-\hat{\mathtt{P}}_0).\mathsf{R}_\gamma.f - (1-\hat{\mathtt{P}}_0).\mathsf{R}_\gamma.\hat{\mathtt{P}}_0.f \\
=& (1-\hat{\mathtt{P}}_0).\mathsf{R}_\gamma.f - (1-\hat{\mathtt{P}}_0).\hat{\mathtt{P}}_0.f \\
=& (1-\hat{\mathtt{P}}_0).\mathsf{R}_\gamma.f
\end{aligned}
$$

proves the lemma. $\square$

Proposition 8.4.14. *The derivative of* $\hat{\Xi} : \mathtt{V}^- \to \mathtt{M}$ *is*

$$d\hat{\Xi} : \mathtt{V}^- \times \mathtt{F}^- \to \mathtt{K}, \quad (f, \Delta\eta) \mapsto d\hat{\Xi}(f, \Delta\eta)$$

with

$$d\hat{\Xi}(f, \Delta\eta) = \hat{\omega}_{\hat{\Gamma}(f)^{-1}}.\hat{\mathtt{P}}_K.\hat{\mathtt{h}}_f.\Delta\eta.$$

Proof.

$$\begin{aligned}
& d\hat{\Xi}(f, \Delta\eta) \\
\overset{8.3.3(iii)}{=} \; & d(\hat{\Pi} \circ \hat{\Gamma})(f, \Delta\eta) \\
= \; & d\hat{\Pi}(\hat{\Gamma}(f), d\hat{\Gamma}(f, \Delta\eta)) \\
\overset{8.1.4}{=} \; & (1 - \hat{\mathtt{P}}_0).d\hat{\Gamma}(f, \Delta\eta) \\
\overset{8.4.12}{=} \; & (1 - \hat{\mathtt{P}}_0).\mathsf{R}_{\hat{\Gamma}(f)}.\hat{\mathtt{P}}_R.(1 - (2\hat{\mathtt{P}}_F + \hat{\mathtt{P}}_0)\hat{\mathtt{P}}_I).\hat{\mathtt{h}}_f.\Delta\eta \\
\overset{8.4.13}{=} \; & (1 - \hat{\mathtt{P}}_0).\mathsf{R}_{\hat{\Gamma}(f)}.(1 - \hat{\mathtt{P}}_0).\hat{\mathtt{P}}_R(1 - (2\hat{\mathtt{P}}_F + \hat{\mathtt{P}}_0)\hat{\mathtt{P}}_I).\hat{\mathtt{h}}_f.\Delta\eta \\
\overset{7.3.3(x)}{=} \; & (1 - \hat{\mathtt{P}}_0).\mathsf{R}_{\hat{\Gamma}(f)}.\hat{\mathtt{P}}_K.\hat{\mathtt{h}}_f.\Delta\eta \\
\overset{8.1.7}{=} \; & \hat{\omega}_{\hat{\Gamma}(f)^{-1}}.\hat{\mathtt{P}}_K.\hat{\mathtt{h}}_f.\Delta\eta.
\end{aligned}$$

□

Remark: So far, we have shown that $d\hat{\Xi}$ is the composition of the the following three functions:

(i)

$$\hat{\mathtt{h}}_f : \mathtt{F}^- \to \mathtt{H}_f, \quad \lambda \mapsto \hat{\mathtt{h}}_f.\lambda$$

with

$$[\hat{\mathtt{h}}_f.\lambda](\tau) = \frac{1}{\hat{\Lambda}(f)'(\tau)} \cdot \lambda(\hat{\Gamma}(f)^{-1}(\tau)),$$

(ii) the projection $\hat{\mathtt{P}}_K : \mathtt{H}_f \to \mathtt{K}$ with $\mathtt{K} = \hat{\mathtt{P}}_R.(1 - \hat{\mathtt{P}}_0).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$, and

(iii)

$$\hat{\omega}_{\hat{\Gamma}(f)-1} : \mathtt{K} \to \mathtt{K}, \quad \lambda \mapsto \lambda \circ \hat{\Gamma}(f) - \int_{\mathbb{S}^1} \lambda(\hat{\Gamma}(f)(\theta))d\theta.$$

8.5 Inverse of $\hat{\mathtt{P}}_K$

In the previous section we have computed the derivative of $\hat{\Xi}$. The result was that $d\hat{\Xi}$ can be written as a chain of three functions. One of these three functions is

$$\hat{\mathtt{P}}_K : \mathtt{H}_f \longrightarrow \mathtt{K}, \quad \lambda \mapsto \hat{\mathtt{P}}_K.\lambda$$

with the domain

$$\mathtt{H}_f := \{ \frac{1}{\hat{\Lambda}(f)'} \cdot \lambda \circ \hat{\Gamma}(f)^{-1} \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \; : \; \lambda \in \mathtt{F}^- \},$$

defined in 8.4.1 and the range

$$\mathtt{K} := \{ \lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}) \; : \; \int_{\mathbb{S}^1} \lambda(t)\, dt = 0 \}$$

defined in 8.1.1, and with

$$(\hat{\mathtt{P}}_K.\lambda)(t) = \cdots + a_{-2}e^{-2it} + a_{-1}e^{-it} + \overline{a}_{-1}e^{it} + \overline{a}_{-2}e^{2it} + \cdots$$

for

$$\lambda(t) = \cdots + a_{-2}e^{-2it} + a_{-1}e^{-it} + a_0 + a_1e^{it} + a_2e^{2it} + \cdots$$

defined in 7.3.2 with the form of Lemma 7.3.7. In this section, we determine the inverse of the linear map $\hat{\mathtt{P}}_K : \mathtt{H}_f \to \mathtt{K}$ explicitly, in order to compute the inverse of the derivative of $\hat{\Xi}$ with respect to the right argument. The strategy is condensed in Lemma 8.5.18. We have to compute the image of the projection $\hat{\mathtt{P}}_K$, $\hat{\mathtt{P}}_{\mathtt{H}_f}$ and their complements. Proposition 8.5.19 gives the result that

$$\hat{\mathtt{P}}_{\mathtt{H}_f} : \mathtt{K} \longrightarrow \mathtt{H}_f$$

is the inverse of $\hat{\mathtt{P}}_K : \mathtt{H}_f \to \mathtt{K}$. The clue of the whole procedure is the function $\hat{\mathtt{P}}_F$ (definition see below), which is used to define $\hat{\mathtt{P}}_{\mathtt{H}_f}$. The main steps are to show that $\hat{\mathtt{P}}_F$ is a projection (Proposition 8.5.6), that $1 - \hat{\mathtt{P}}_F$ projects onto $\mathtt{F}^+$ (Proposition 8.5.12), that $1 - \hat{\mathtt{P}}_{\mathtt{H}_f}$ projects also onto $\mathtt{F}^+$ (Lemma 8.5.14), and that $\hat{\mathtt{P}}_{\mathtt{H}_f}$ projects onto $\mathtt{H}_f$ (Lemma 8.5.17).

Definition 8.5.1. For $f \in \mathtt{V}$ we define the projection

$$\hat{\mathtt{P}}_f : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \lambda \mapsto \hat{\mathtt{P}}_f.\lambda$$

by

$$(\hat{\mathtt{P}}_f.\lambda)(\tau) := \frac{i}{2\pi} \int_{\mathbb{S}^1} \frac{\lambda(\theta) - \lambda(\tau)}{f(\theta) - f(\tau)} f'(\theta) d\theta.$$

Remark: In Proposition 5.4.4, we have already shown that

$$\mathtt{V} \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad (f, \lambda) \mapsto \hat{\mathtt{P}}_f.\lambda$$

is smooth-tame.

Definition 8.5.2. Let $f \in \mathtt{V}$. We define the linear map

$$\hat{\mathtt{P}}_{\mathtt{H}_f} : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \lambda \mapsto \hat{\mathtt{P}}_{\mathtt{H}_f}.\lambda$$

by

$$(\hat{\mathtt{P}}_{\mathtt{H}_f}.\lambda)(t) = \frac{\hat{\Lambda}(f)(t)}{\hat{\Lambda}(f)'(t)}[\hat{\mathtt{P}}_{\hat{\Lambda}(f)}.\{\theta \mapsto \lambda(\theta) \cdot \frac{\hat{\Lambda}(f)'(\theta)}{\hat{\Lambda}(f)(\theta)}\}](t).$$

Remark: The restriction

$$\hat{\mathtt{P}}_{\mathtt{H}_f} : \mathtt{K} \longrightarrow \mathtt{H}_f$$

is our candidate for the inverse of $\hat{\mathtt{P}}_K : \mathtt{H}_f \to \mathtt{K}$. To verify this, we must show that

$$\begin{aligned} (1 - \hat{\mathtt{P}}_{\mathtt{H}_f}).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) &= (1 - \hat{\mathtt{P}}_K).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \\ \hat{\mathtt{P}}_K.\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) &= \mathtt{K}, \quad \text{and} \\ \hat{\mathtt{P}}_{\mathtt{H}_f}.\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) &= \mathtt{H}_f \end{aligned}$$

holds and use some linear algebra arguments (Lemma 8.5.18). We do this in the rest of the section.

Definition 8.5.3. Let $f \in \mathtt{V}$ and $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$. We define the function

$$\hat{\Omega}_f.\lambda : \overline{\mathtt{D}}_f \to \mathbb{C}, \quad z \mapsto (\hat{\Omega}_f.\lambda)(z)$$

by

$$(\hat{\Omega}_f.\lambda)(z) := \frac{1}{2\pi i}\int_0^{2\pi} \frac{\lambda(t)}{f(t) - z} \cdot f'(t)dt$$

for $z \in \mathtt{D}_f$, and by

$$[\hat{\Omega}_f.\lambda](z) := [(1 - \hat{\mathtt{P}}_f).\lambda](f^{-1}(z))$$

for $z \in \partial\mathtt{D}_f$. Furthermore, in the case of $f(t) = e^{it}$ we write $\hat{\Omega}_E$ instead of $\hat{\Omega}_f$.

Lemma 8.5.4. *Let $f \in \mathtt{V}$ and $\xi \in (-\mathtt{r}_f, 0)$, then*

$$\mathtt{N}_f(\xi, \theta) := f(\theta) - i\frac{f'(\theta)}{|f'(\theta)|}\xi \in \mathtt{D}_f.$$

Here $\mathtt{r}_f$ denotes the tube radius defined in 6.1.1.

Proof. By Lemma 6.3.3, the subset $\mathtt{V}$ is path connected. This means, there exists a continuous function

$$H : [0,1] \to \mathtt{V}, \quad \lambda \mapsto H(\lambda)$$

such that $H(0) = p$ and $H(1) = f$ holds, with

$$p : \mathbb{S}^1 \to \mathbb{C}, \quad t \mapsto e^{it}.$$

Moreover, the tube radius function $R : \mathtt{V} \to \mathbb{R}^+$, $f \mapsto \mathtt{r}_f$ is continuous, what can be obtained form the Definition 6.1.1. This implies that

$$r := \mathsf{min}\{\, R(H(\lambda)) \in \mathbb{R}^+ \,:\, \lambda \in [0,1]\,\} > 0,$$

because $[0,1]$ is compact. Since $r \leq \mathtt{r}_{H(\lambda)}$ the function

$$\mathtt{N}_{H(\lambda)} : (-r, 0] \times \mathbb{S}^1, \quad (\xi, \theta) \mapsto \mathtt{N}_{H(\lambda)}(\xi, \theta)$$

is injective by Lemma 6.1.5, which implies

$$H(\lambda) - \mathtt{N}_{H(\lambda)} \neq 0$$

for all $t, \theta \in \mathbb{S}^1$ and $\xi \in (-r, 0)$. Now fix an element $\xi \in (-r, 0)$ and an element $\theta \in \mathbb{S}^1$ for a moment. The function

$$B : \mathtt{V} \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times), \quad f \mapsto f - \mathtt{N}_f(\xi, \theta)$$

is continuous, and $B(H(\lambda)) \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times)$ for all $\lambda \in [0,1]$. This shows that the function

$$\mathtt{w} \circ B \circ H : [0,1] \to \mathbb{Z}$$

is also continuous. Due to 6.2.5 we have

$$\mathtt{N}_p(\xi, \theta) = e^{i\theta}(1 + \xi),$$

and therefore

$$\mathtt{w}(B(H(0))) = \mathtt{w}(B(p)) = \frac{1}{2\pi}\int_{\mathbb{S}^1} \frac{ie^{it}}{e^{it} - e^{i\theta}(1+\xi)}dt = 1.$$

Hence, $\mathtt{w} \circ B \circ H \equiv 1$. In particular, we have $\mathtt{w}(H(B(1))) = 1$. Writing this explicitly

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} \frac{f'(t)}{f(t) - \mathtt{N}_f(\xi, \theta)} dt = 1,$$

we obtain $\mathtt{N}_f(\xi, \theta) \in \mathtt{D}_f$ by Definition 2.4.4. So far, we have shown the lemma for the case $\xi \in (-r, 0)$. The function

$$\mathtt{N}_f : (-\mathtt{r}_f, 0) \times \mathbb{S}^1 \to \mathbb{C}$$

is smooth, which implies that

$$(-\mathtt{r}_f, 0) \times \mathbb{S}^1 \to \mathbb{Z}, \quad (\xi, \theta) \mapsto \frac{1}{2\pi} \int_{\mathbb{S}^1} \frac{f'(t)}{f(t) - \mathtt{N}_f(\xi, \theta)} dt,$$

is continuous. In particular we have

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} \frac{f'(t)}{f(t) - \mathtt{N}_f(\xi, \theta)} dt = 1,$$

for all $\xi \in (-\mathtt{r}_f, 0)$ and $\theta \in \mathbb{S}^1$. By Definition 2.4.4, we get $\mathtt{N}_f(\xi, \theta) \in \mathtt{D}_f$ for all $\xi \in (-\mathtt{r}_f, 0)$ and $\theta \in \mathbb{S}^1$, which is the statement of the lemma. $\square$

Lemma 8.5.5. *If $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ then the function $\hat{\Omega}_f.\lambda : \overline{\mathtt{D}}_f \to \mathbb{C}$ with $f \in \mathtt{V}$ is*

(i) holomorphic on $\mathtt{D}_f$, and

(ii) continuous on $\overline{\mathtt{D}}_f$.

Proof. (i): Let $z, z_0 \in \mathtt{D}_f$, then Definition 8.5.3 yields

$$\begin{aligned}
(\hat{\Omega}_f.\lambda)(z) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{\lambda(t)}{f(t) - z} \cdot f'(t) dt \\
&= \frac{1}{2\pi i} \int_0^{2\pi} [\frac{1}{f(t) - z_0} + \frac{z - z_0}{(f(t) - z)(f(t) - z_0)}] \lambda(t) f'(t) dt \\
&= (\hat{\Omega}_f.\lambda)(z_0) + (z - z_0) h(z)
\end{aligned}$$

with

$$h(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\lambda(t) \cdot f'(t)}{(f(t) - z)(f(t) - z_0)} dt.$$

The function $h : \mathtt{D}_f \to \mathbb{C}, \quad z \mapsto h(z)$ is continuous, because its integrand depends continuously on z. Hence, the statement is proved.
(ii): Recall that $\mathtt{r}_f > 0$ denotes the tube radius defined in 6.1.1, and

$$\mathtt{N}_f(\xi, \theta) := f(\theta) - i \frac{f'(\theta)}{|f'(\theta)|} \cdot \xi$$

the tube function defined in 6.1.3.
1. Step: Defining G:
Let $f \in \mathtt{V}$ and $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$. We define

$$G_\epsilon(\xi,\theta) = \frac{1}{2\pi i}\int_{\theta+\epsilon}^{\theta+2\pi-\epsilon} \frac{f(\theta) - \mathtt{N}_f(\xi,\theta)}{f(t) - \mathtt{N}_f(\xi,\theta)} \cdot \frac{\lambda(t)-\lambda(\theta)}{f(t)-f(\theta)} \cdot f'(t)dt$$

for $\epsilon > 0$, $|\xi| < \mathtt{r}_f$ and $\theta \in \mathbb{S}^1$. We will show that the limit

$$G(\xi,\theta) := \mathsf{lim}_{\epsilon\to 0} G_\epsilon(\xi,\theta)$$

exists for $\xi \in (-\mathtt{r}_f, \mathtt{r}_f)$ and $\theta \in \mathbb{S}^1$, and that G defines a continuous function. Lemma 6.1.6 asserts

$$|f(t) - \mathtt{N}_f(\xi,\theta)| \geq \xi$$

for $t, \theta \in \mathbb{S}^1$ and $|\xi| < \mathtt{r}_f$. Taking

$$|f(\theta) - \mathtt{N}_f(\xi,\theta)| = |f(\theta) - f(\theta) + i\frac{f'(\theta)}{|f'(\theta)|} \cdot \xi| = |\xi|$$

into account, this implies

$$\frac{|f(\theta) - \mathtt{N}_f(\xi,\theta)|}{|f(t) - \mathtt{N}_f(\xi,\theta)|} \leq 1 \qquad (\dagger)$$

for $0 < |\xi| < \mathtt{r}_f$ and $t, \theta \in \mathbb{S}$. On the other hand, if $\xi = 0$ and $t \neq \theta$, we have

$$|f(t) - \mathtt{N}_f(\xi,\theta)| = |f(t) - f(\theta) + i\frac{f'(\theta)}{|f'(\theta)|} \cdot \xi| = |f(t) - f(\theta)| \neq 0$$

because f is injective, and in addition $|f(\theta) - \mathtt{N}_f(\xi,\theta)| = 0$. This implies that $(\dagger)$ holds for $t \neq \theta$ and all $\xi \in \mathbb{R}$ with $|\xi| < \mathtt{r}_f$. Moreover, $t, \theta \in \mathbb{S}^1$ the term

$$\frac{\lambda(t)-\lambda(\theta)}{f(t)-f(\theta)} \cdot f'(t)$$

is bounded, because it depends smoothly on t and θ, and $\mathbb{S}^1 \times \mathbb{S}^1$ is compact. So far, we obtain that for all $\theta, t \in \mathbb{S}^1$ and $|\xi| < \mathtt{r}_f$ the integrand

$$\frac{f(\theta) - \mathtt{N}_f(\xi,\theta)}{f(t) - \mathtt{N}_f(\xi,\theta)} \cdot \frac{\lambda(t)-\lambda(\theta)}{f(t)-f(\theta)} \cdot f'(t)$$

is bounded almost everywhere (except for $\xi = 0$ and $t = \theta$) by a real constant $C > 0$. Hence, we can define

$$G(\xi, \theta) := \mathsf{lim}_{\epsilon \to 0} G_\epsilon(\xi, \theta)$$

for $\xi \in (-\mathtt{r}_f, \mathtt{r}_f)$ and $\theta \in \mathbb{S}^1$. In particular, we have

$$|G_{1/n}(\xi, \theta) - G(\xi, \theta)| \leq \frac{1}{n} \cdot \frac{C}{2\pi}$$

for all $\xi \in (-\mathtt{r}_f, \mathtt{r}_f)$ and $\theta \in \mathbb{S}^1$. This means, that the series $(G_{1/n})_{n \in \mathbb{N}}$ converges uniformly to the function G. Taking into account that $G_{1/n}$ depends continuously on ξ and θ, we conclude that the function

$$G : (-\mathtt{r}_f, \mathtt{r}_f) \times \mathbb{S}^1 \to \mathbb{C}, \quad (\xi, \theta) \mapsto G(\xi, \theta)$$

is continuous. 2. Step:

$$(\hat{\Omega}_f.\lambda)(\mathtt{N}_f(\xi, \theta)) = [(1 - \hat{\mathtt{P}}_f).\lambda](\theta) - G(\xi, \theta)$$

for $\xi \in (-\mathtt{r}_f, 0]$ and $\theta \in \mathbb{S}^1$:
First, we consider the case $\xi = 0$ and $\theta \in \mathbb{S}^1$. We have $f(\theta) - \mathtt{N}_f(0, \theta) = 0$, which implies $G_\epsilon(0, \theta) = 0$ and hence $G(0, \theta) = 0$. We compute

$$\begin{aligned}
(\hat{\Omega}_f.\lambda)(\mathtt{N}_f(0, \theta)) &= (\hat{\Omega}_f.\lambda)(f(\theta)) \\
&\overset{8.5.3}{=} [(1 - \hat{\mathtt{P}}_f).\lambda](\theta) \\
&= [(1 - \hat{\mathtt{P}}_f).\lambda](\theta) + G(0, \theta).
\end{aligned}$$

Secondly, we consider the case $\xi \in (-\mathtt{r}_f, 0)$ and $\theta \in \mathbb{S}^1$. Lemma 8.5.4 asserts that $\mathtt{N}_f(\xi, \theta) \in \mathtt{D}_f$ and by Definition 2.4.4 we have

$$\frac{1}{2\pi i} \int_0^{2\pi} \frac{f'(t)}{f(t) - \mathtt{N}_f(\xi, \theta)} dt = 1. \qquad (\dagger\dagger)$$

So, we compute

$$
\begin{aligned}
&(\hat{\Omega}_f.\lambda)(\mathtt{N}_f(\xi,\theta)) \\
\overset{8.5.3}{=}\ & \frac{1}{2\pi i}\int_0^{2\pi} \frac{\lambda(t)}{f(t)-\mathtt{N}_f(\xi,\theta)} f'(t)dt \\
=\ & \lambda(\theta)\frac{1}{2\pi i}\int_0^{2\pi} \frac{f'(t)}{f(t)-\mathtt{N}_f(\xi,\theta)}dt + \frac{1}{2\pi i}\int_0^{2\pi} \frac{\lambda(t)-\lambda(\theta)}{f(t)-\mathtt{N}_f(\xi,\theta)} f'(t)dt \\
\overset{(\dagger\dagger)}{=}\ & \lambda(\theta) + \frac{1}{2\pi i}\int_0^{2\pi} \frac{\lambda(t)-\lambda(\theta)}{f(t)-\mathtt{N}_f(\xi,\theta)} f'(t)dt \\
=\ & \lambda(\theta) + \frac{1}{2\pi i}\int_0^{2\pi} \frac{\lambda(t)-\lambda(\theta)}{f(t)-f(\theta)} f'(t)dt \\
& - \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\theta)-\mathtt{N}_f(\xi,\theta)}{f(t)-\mathtt{N}_f(\xi,\theta)} \cdot \frac{\lambda(t)-\lambda(\theta)}{f(t)-f(\theta)} \cdot f'(t)dt \\
\overset{8.5.1}{=}\ & \lambda(\theta) - [\hat{\mathtt{P}}_f.\lambda](\theta) - G(\xi,\theta) \\
=\ & [(1-\hat{\mathtt{P}}_f).\lambda](\theta) - G(\xi,\theta).
\end{aligned}
$$

3. Step: $\hat{\Omega}_f.\lambda : U \to \mathbb{C}$ is continuous:
Define $U := \{\, \mathtt{N}_f(\xi,\theta) \in \mathbb{C} \,:\, \xi \in (-\mathtt{r}_f, 0] \wedge \theta \in \mathbb{S}^1 \,\}$. By the second step of this proof, we have

$$(\hat{\Omega}_f.\lambda)(z) = [(1-\hat{\mathtt{P}}_f)](\pi_2(\mathtt{N}_f^{-1}(z))) - G(\mathtt{N}_f^{-1}(z))$$

for $z \in U$. Moreover, the three functions

$$
\begin{aligned}
\mathtt{N}_f^{-1} &: U \to (-\mathtt{r}_f, 0] \times \mathbb{S}^1, \\
(1-\hat{\mathtt{P}}_f).\lambda &: \mathbb{S}^1 \to \mathbb{C}, \qquad \text{and} \\
G &: (-\mathtt{r}_f, 0] \times \mathbb{S}^1 \to \mathbb{C} \qquad \text{[by the 1. Step]}
\end{aligned}
$$

are continuous, and hence the function $\hat{\Omega}_f.\lambda : U \to \mathbb{C}$ is continuous, too.
4. Step: $\hat{\Omega}_f.\lambda : \overline{\mathtt{D}}_f \to \mathbb{C}$ is continuous:
Consider the subset $\overline{\mathtt{D}}_f \subseteq \mathbb{C}$ with the induced topology. The subsets $\mathtt{D}_f$ and U are open in $\overline{\mathtt{D}}_f$ and we have the covering $\overline{\mathtt{D}}_f = \mathtt{D}_f \cup U$. The map $(\hat{\Omega}_f.\lambda)$ is continuous on U by the third Step and is continuous on $\mathtt{D}_f$ by part (i). Hence, $\hat{\Omega}_f.\lambda$ is continuous on $\overline{\mathtt{D}}_f = \mathtt{D}_f \cup U$. $\square$

Proposition 8.5.6. *If* $f \in \mathtt{V}$, *then* $\hat{\mathtt{P}}_f : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ *is a projection.*

Proof. By Lemma 8.5.5, the function $(\hat{\Omega}_f.\lambda)$ is continuous on $\overline{\mathtt{D}}_f$ and holomorphic on the interior $\mathtt{D}_f$. Due to the Cauchy Theorem, we can express the value of $(\hat{\Omega}_f.\lambda)$ on $\mathtt{D}_f$ via an integral along the curve $\partial\mathtt{D}_f$ as follows.

$$\begin{aligned}(\hat{\Omega}_f.\lambda)(\mathtt{N}_f(\xi,\theta)) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{(\hat{\Omega}_f.\lambda)(f(t))}{f(t) - \mathtt{N}_f(\xi,\theta)} f'(t)dt \\ &\overset{8.5.3}{=} [\hat{\Omega}_f.(1 - \hat{\mathtt{P}}_f).\lambda](\mathtt{N}_f(\xi,\theta)).\end{aligned}$$

By Lemma 8.5.5(ii), the function $\hat{\Omega}_f.\lambda$ is continuous on $\overline{\mathtt{D}}_f$, such that we can take the limit on both sides

$$\lim_{\xi\to 0} (\hat{\Omega}_f.\lambda)(\mathtt{N}_f(\xi,\theta)) = \lim_{\xi\to 0} [\hat{\Omega}_f.(1 - \hat{\mathtt{P}}_f).\lambda](\mathtt{N}_f(\xi,\theta)),$$

i.e.,

$$(\hat{\Omega}_f.\lambda)(f(\theta)) = [\hat{\Omega}_f.(1 - \hat{\mathtt{P}}_f).\lambda](f(\theta)),$$

and get

$$(1 - \hat{\mathtt{P}}_f).\lambda = (1 - \hat{\mathtt{P}}_f).(1 - \hat{\mathtt{P}}_f).\lambda$$

according to Definition 8.5.3. Hence, we have shown that $1 - \hat{\mathtt{P}}_f$ is a projection, and finally $\hat{\mathtt{P}}_f$ is a projection too. □

Lemma 8.5.7. *If* $f \in \mathtt{V}$, *then*
(i) $\hat{\mathtt{P}}_f.f^n = 0$ *for* $n \geq 0$, *and*
(ii) $\hat{\mathtt{P}}_f.\frac{1}{f^n} = \frac{1}{f^n}$ *for* $n > 0$.

Proof. (i) For the case $n = 0$ the function f^0 is the constant function such that $\hat{\mathtt{P}}_f.f^n = 0$. If $n > 0$, then

$$\begin{aligned}[\hat{\mathtt{P}}_f.f^n](\tau) &= \frac{i}{2\pi}\int_{\mathbb{S}^1} \frac{f^n(\theta) - f^n(\tau)}{f(\theta) - f(\tau)} f'(\theta)d\theta \\ &= \frac{i}{2\pi}\sum_{j=0}^{n-1}\int_{\mathbb{S}^1} f^j(\theta)f^{n-1-j}(\tau)f'(\theta)d\theta \\ &= \frac{i}{2\pi}\sum_{j=0}^{n-1} f^{n-1-j}(\tau)\underbrace{[\frac{1}{j+1}f^{j+1}(\theta)]_{\theta=0}^{\theta=2\pi}}_{=0} = 0.\end{aligned}$$

(ii)

$$
\begin{aligned}
[\hat{\mathtt{P}}_f.\frac{1}{f^n}](\tau) &= \frac{i}{2\pi}\int_{\mathbb{S}^1}\frac{\frac{1}{f^n(\theta)}-\frac{1}{f^n(\tau)}}{f(\theta)-f(\tau)}f'(\theta)d\theta \\
&= -\frac{i}{2\pi}\frac{1}{f^n(\tau)}\int_{\mathbb{S}^1}\frac{f^n(\theta)-f^n(\tau)}{f(\theta)-f(\tau)}\frac{f'(\theta)}{f^n(\theta)}d\theta \\
&= \frac{1}{2\pi i}\frac{1}{f^n(\tau)}\sum_{j=0}^{n-1}\int_{\mathbb{S}^1} f^j(\tau)f^{n-1-j}(\theta)\frac{f'(\theta)}{f^n(\theta)}d\theta \\
&= \frac{1}{f^n(\tau)}\underbrace{\frac{1}{2\pi i}\int_{\mathbb{S}^1}\frac{f'(\theta)}{f(\theta)}d\theta}_{=\mathtt{w}(f)=1} \\
&= \frac{1}{f^n(\tau)}.
\end{aligned}
$$

□

Lemma 8.5.8. *If $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C})$, then*

(i) $(\hat{\Omega}_E.\lambda)(z) = \frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{\lambda(t)}{1-ze^{-it}}dt$ *for* $z\in\mathbb{D}$, *and*

(ii) $(\hat{\Omega}_E.\lambda)(e^{it}) = [(\hat{\mathtt{P}}_0+\hat{\mathtt{P}}_F).\lambda](t)$ *for* $t\in\mathbb{R}$.

Proof. By Definition 8.5.3, the function $\hat{\Omega}_E$ is the special case of $\hat{\Omega}_f$ for $f(t)=e^{it}$. Throughout this proof, we have $f(t)=e^{it}$ and $\mathtt{D}_f=\mathbb{D}$.
(i):

$$
\begin{aligned}
[\hat{\Omega}_f.\lambda](z) &\overset{8.5.3}{=} \frac{1}{2\pi i}\int_0^{2\pi}\frac{\lambda(t)}{f(t)-z}\cdot f'(t)dt \\
&= \frac{1}{2\pi i}\int_0^{2\pi}\frac{\lambda(t)}{e^{it}-z}\cdot ie^{it}dt \\
&= \frac{1}{2\pi}\int_{\mathbb{S}^1}\frac{\lambda(t)}{1-ze^{-it}}dt.
\end{aligned}
$$

(ii):

$$\begin{aligned}
[\hat{\Omega}_f.\lambda](e^{it}) &\overset{8.5.3}{=} [(1-\hat{\mathtt{P}}_f).\lambda](f^{-1}(e^{it})) \\
&= [(1-\hat{\mathtt{P}}_f).\lambda](t) \\
&\overset{8.5.1}{=} \lambda(t) - \frac{i}{2\pi}\int_{\mathbb{S}^1} \frac{\lambda(\theta)-\lambda(t)}{f(\theta)-f(t)} f'(\theta)d\theta \\
&= \lambda(t) - \frac{i}{2\pi}\int_{\mathbb{S}^1} \frac{\lambda(\theta)-\lambda(t)}{e^{i\theta}-e^{it}} ie^{i\theta}d\theta \\
&= \lambda(t) - \frac{1}{2\pi}\int_{\mathbb{S}^1} \frac{\lambda(t)-\lambda(\theta)}{1-e^{i(t-\theta)}} d\theta \\
&= \lambda(t) - \frac{1}{2\pi}\int_{\mathbb{S}^1} \left(1 - \frac{1}{1-e^{-i(t-\theta)}}\right)(\lambda(t)-\lambda(\theta)\,)d\theta \\
&= \frac{1}{2\pi}\int_{\mathbb{S}^1} \left(\lambda(\theta) + \frac{\lambda(t)-\lambda(\theta)}{1-e^{-i(t-\theta)}}\right) d\theta \\
&\overset{7.3.6(ii)}{=} (\hat{\mathtt{P}}_0.\lambda)(t) + \frac{1}{2\pi}\int_{\mathbb{S}^1} \left(\frac{\lambda(t)-\lambda(\theta)}{1-e^{-i(t-\theta)}}\right) d\theta \\
&\overset{7.3.8}{=} [(\hat{\mathtt{P}}_0 + \hat{\mathtt{P}}_F).\lambda](t).
\end{aligned}$$

□

Lemma 8.5.9. *Let $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$. The Taylor series of $\hat{\Omega}_E.\lambda$ at zero is*

$$(\hat{\Omega}_E.\lambda)(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

for $z \in \overline{\mathbb{D}}$, and the coefficients are

$$a_n = \frac{1}{2\pi}\int_{\mathbb{S}^1} e^{-int}\lambda(t)dt$$

for $n \in \mathbb{N}_0$.

Proof. Lemma 8.5.8(i) yields

$$(\hat{\Omega}_E.\lambda)(z) = \frac{1}{2\pi}\int_{\mathbb{S}^1} \frac{\lambda(t)}{1-ze^{-it}} dt$$

for $z \in \mathbb{D}$, its n-th derivative with respect to z is

$$(\hat{\Omega}_E.\lambda)^{(n)}(z) = \frac{n!}{2\pi}\int_{\mathbb{S}^1} e^{-int}\frac{\lambda(t)}{(1-ze^{-it})^n} dt.$$

After the substitution $z = 0$, we get the Taylor coefficients by

$$\frac{1}{n!}(\hat{\Omega}_E.\lambda)^{(n)}(0) = \frac{1}{2\pi}\int_{\mathbb{S}^1} e^{-int}\lambda(t)dt.$$

This shows that the Taylor coefficients $(\hat{\Omega}_E.\lambda)$ are the Fourier coefficients (with positive index) of the smooth function $\lambda : \mathbb{S}^1 \to \mathbb{C}$. Lemma 2.2.4(ii) guarantees

$$\sum_{n\in\mathbb{N}_0} |a_n| < \infty,$$

and we obtain that the Taylor series of $(\hat{\Omega}_E.\lambda)$ converges absolutely on $\overline{\mathbb{D}}$.

□

Lemma 8.5.10. *If $\lambda \in \mathtt{F}^+$, then $(\hat{\Omega}_E.\lambda)(e^{it}) = \lambda(t)$ for every $t \in \mathbb{S}^1$.*

Proof. We remember from Definition 2.2.3 that $\mathtt{F}^+ \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ is the subspace of all functions of the form

$$\lambda(t) = a_0 + a_1 e^{it} + a_2 e^{2it} + \cdots ,$$

and we obtain

$$(\hat{\Omega}_E.\lambda)(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

by Lemma 8.5.9. Since the function

$$(\hat{\Omega}_E.\lambda) : \overline{\mathbb{D}} \to \mathbb{C}, \quad z \mapsto (\hat{\Omega}_E.\lambda)(z)$$

is continuous by Lemma 8.5.5, the substitution $z = e^{it}$ yields

$$(\hat{\Omega}_E.\lambda)(e^{it}) = a_0 + a_1 e^{it} + a_2 e^{2it} + \cdots ,$$

and we conclude

$$(\hat{\Omega}_E.\lambda)(e^{it}) = \lambda(t).$$

□

Lemma 8.5.11. *Let $f \in \mathtt{V}^E$, then the map*

$$\hat{\Omega}_E.f : \overline{\mathbb{D}} \to \overline{\mathtt{D}}_f$$

is univalent on $\mathbb{D}$, extends diffeomorphically to the boundary, and satisfies

$$(\hat{\Omega}_E.f)(e^{it}) = f(t).$$

Proof. By Definition 2.4.3, the Fourier series of $f \in \mathtt{V}^E$ is

$$f(t) = e^{it}(r + a_1 e^{it} + a_2 e^{2it} + \cdots),$$

and Lemma 8.5.9 yields

$$(\hat{\Omega}_E.f)(z) = z(r + a_1 z + a_2 z^2 + \cdots)$$

for $z \in \overline{\mathbb{D}}$. Using Lemma 2.4.6, we obtain that $\hat{\Omega}_E.f$ is a biholomorphic map from $\mathbb{D}$ onto $\mathtt{D}_f$, and by Lemma 8.5.10 we conclude

$$(\hat{\Omega}_E.f)(e^{it}) = f(t).$$

$\square$

Proposition 8.5.12. *If $f \in \mathtt{V}^E$, then $(1 - \hat{\mathtt{P}}_f).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) = \mathtt{F}^+$. In other words, $(1 - \hat{\mathtt{P}}_f)$ is a projection onto $\mathtt{F}^+$.*

Proof. 1. Step: $(1 - \hat{\mathtt{P}}_f).\lambda = \lambda$ for $\lambda \in \mathtt{F}^+$:
Fix two elements $\lambda \in \mathtt{F}^+$ and $f \in \mathtt{V}^E$. By Lemma 8.5.11, the function

$$\hat{\Omega}_E.f : \overline{\mathbb{D}} \to \overline{\mathtt{D}}_f$$

is bijective and univalent on the interior $\mathbb{D}$, and by Lemma 8.5.9 the function

$$\hat{\Omega}_E.\lambda : \overline{\mathbb{D}} \to \mathbb{C}$$

is holomorphic on the interior $\mathbb{D}$. Therefore, the function

$$L : \overline{\mathtt{D}}_f \to \mathbb{C}, \quad z \mapsto (\hat{\Omega}_E.\lambda)((\hat{\Omega}_E.f)^{-1}(z))$$

is holomorphic on $\mathtt{D}_f$ and continuous on $\overline{\mathtt{D}}_f$. Moreover, we have

$$\begin{aligned} L(f(\tau)) &= (\hat{\Omega}_E.\lambda)((\hat{\Omega}_E.f)^{-1}(f(\tau))) \\ &\overset{8.5.11}{=} (\hat{\Omega}_E.\lambda)(e^{i\tau}) \\ &\overset{8.5.10}{=} \lambda(\tau) \qquad\qquad (\dagger) \end{aligned}$$

for all $\tau \in \mathbb{S}^1$, and hence

$$\begin{aligned}
& [(1-\hat{\mathtt{P}}_f).\lambda](\theta) \\
\overset{8.5.3}{=}\ & (\hat{\Omega}_f.\lambda)(f(\theta)) \\
\overset{6.1.3}{=}\ & (\hat{\Omega}_f.\lambda)(\mathtt{N}_f(0,\theta)) \\
\overset{8.5.5}{=}\ & \mathsf{lim}_{\xi<0,\xi\to 0}(\hat{\Omega}_f.\lambda)(\mathtt{N}_f(\xi,\theta)) \\
\overset{8.5.3}{=}\ & \mathsf{lim}_{\xi<0,\xi\to 0}\frac{1}{2\pi i}\int_0^{2\pi}\frac{\lambda(t)}{f(t)-\mathtt{N}_f(\xi,\theta)}f'(t)dt \\
\overset{(\dagger)}{=}\ & \mathsf{lim}_{\xi<0,\xi\to 0}\frac{1}{2\pi i}\int_0^{2\pi}\frac{L(f(t))}{f(t)-\mathtt{N}_f(\xi,\theta)}f'(t)dt \\
=\ & \mathsf{lim}_{\xi<0,\xi\to 0}L(\mathtt{N}_f(\xi,\theta)) \qquad \text{[By the Cauchy Theorem]} \\
=\ & L(\mathtt{N}_f(0,\theta)) \\
\overset{6.1.3}{=}\ & L(f(\theta)) \\
\overset{(\dagger)}{=}\ & \lambda(\theta)
\end{aligned}$$

for all $\theta \in \mathbb{S}^1$.
2. Step: $(1-\hat{\mathtt{P}}_f).\mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \subseteq \mathtt{F}^+$:
Let $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C})$ and $f \in \mathtt{V}^E$. By Lemma 8.5.11, the function

$$(\hat{\Omega}_E.f) : \overline{\mathbb{D}} \to \overline{\mathtt{D}}_f$$

is continuous on $\overline{\mathbb{D}}$ and holomorphic on $\mathbb{D}$. By Lemma 8.5.5, the function

$$\hat{\Omega}_f.\lambda : \overline{\mathtt{D}}_f \to \mathbb{C}$$

is continuous on $\overline{\mathtt{D}}_f$ and holomorphic on $\mathtt{D}_f$. Therefore, the composition

$$(\hat{\Omega}_f.\lambda) \circ (\hat{\Omega}_E.f) : \overline{\mathbb{D}} \to \mathbb{C}$$

is continuous on $\overline{\mathbb{D}}$ and holomorphic on $\mathbb{D}$, which allows the last step of the computation

$$\begin{aligned}
[(1-\hat{\mathtt{P}}_f).\lambda](t) &\overset{8.5.3}{=} (\hat{\Omega}_f.\lambda)(f(t)) \\
&\overset{8.5.10}{=} (\hat{\Omega}_f.\lambda)(\hat{\Omega}_E.f(e^{it})) \\
&= a_0 + a_1 e^{it} + a_2 e^{it} + \cdots
\end{aligned}$$

for all $t \in \mathbb{S}^1$. Hence, we have $(1-\hat{\mathtt{P}}_F).\lambda \in \mathtt{F}^+$.
3. Step:
By Proposition 8.5.6, the map $\hat{\mathtt{P}}_f : \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C})$ is a projection and therefore $1-\hat{\mathtt{P}}_f$ is a projection, too. With the first and second step we conclude that $1-\hat{\mathtt{P}}_f$ projects onto $\mathtt{F}^+$. $\square$

Lemma 8.5.13. *If $f \in V$, then*

(i) $\dfrac{\hat{\Lambda}(f)(t)}{\hat{\Lambda}(f)'(t)} = a_0 + a_1 e^{it} + a_2 e^{2it} + \cdots$, *and*

(ii) $\frac{\hat{\Lambda}(f)'(t)}{\hat{\Lambda}(f)(t)} = b_0 + b_1 e^{it} + b_2 e^{2it} + \cdots$ *for all* $t \in \mathbb{S}^1$.

Proof. The Fourier series of the injective function $\hat{\Lambda}(f) \in \mathtt{V}^E$ is

$$\hat{\Lambda}(f)(t) = e^{it}(r + a_1 e^{it} + a_2 e^{2it} + \cdots).$$

By Lemma 2.4.6, the function $F : \overline{\mathbb{D}} \to \overline{\mathtt{D}}_{\hat{\Lambda}(f)}$ defined by

$$F(z) = z(r + a_1 z + a_2 z^2 + \cdots)$$

is biholomorphic on $\mathbb{D}$ and fulfills $F(0) = 0$. Since F is injective, we have

$$\frac{F(z)}{z} \neq 0$$

for all $z \in \overline{\mathbb{D}}$. Furthermore, $F'(z) \neq 0$ holds for all $z \in \overline{\mathbb{D}}$, because F is biholomorphic. Hence,

$$G(z) = iz\frac{F'(z)}{F(z)}$$

is a holomorphic function on $\mathbb{D}$ and never zero for any $z \in \overline{\mathbb{D}}$. The same holds for $1/G(z)$. Finally, we conclude that

$$\frac{\hat{\Lambda}(f)(t)}{\hat{\Lambda}(f)'(t)} = \frac{1}{G(e^{it})}$$

and

$$\frac{\hat{\Lambda}(f)'(t)}{\hat{\Lambda}(f)(t)} = G(e^{it}),$$

such that the Lemma is proved. □

Lemma 8.5.14. *If $f \in \mathtt{V}$, then $(1 - \hat{\mathtt{P}}_{\mathtt{H}_f}).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) = \mathtt{F}^+$.*

Proof. 1. Step: $(1 - \hat{\mathtt{P}}_{\mathtt{H}_f}).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \subseteq \mathtt{F}^+$:
Lemma 8.5.12 yields

$$[(1 - \hat{\mathtt{P}}_{\hat{\Lambda}(f)}).\lambda](t) = b_0 + b_1 e^{it} + b_2 e^{2it} + \cdots$$

for all $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$. By Lemma 8.5.13(i), we have

$$\frac{\hat{\Lambda}(f)(t)}{\hat{\Lambda}(f)'(t)} = a_0 + a_1 e^{it} + a_2 e^{2it} + \cdots$$

and get

$$\frac{\hat{\Lambda}(f)(t)}{\hat{\Lambda}(f)'(t)} \cdot [(1 - \hat{\mathtt{P}}_{\hat{\Lambda}(f)}).\lambda](t) = c_0 + c_1 e^{it} + \cdots$$

for every $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$. Hence, we have

$$\begin{aligned} & (1 - \hat{\mathtt{P}}_{\mathtt{H}_f}).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \\ = \ & \{ \frac{\hat{\Lambda}(f)}{\hat{\Lambda}(f)'} \cdot [(1 - \hat{\mathtt{P}}_{\hat{\Lambda}(f)}).\lambda] \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \ : \ \lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \} \subseteq \mathtt{F}^+ \end{aligned}$$

by the Definition 8.5.2 of $\hat{\mathtt{P}}_{\mathtt{H}_f}$.
2. Step: $\mathtt{F}^+ \subseteq (1 - \hat{\mathtt{P}}_{\mathtt{H}_f}).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$:
Let $g \in \mathtt{F}^+$ be a function with a Fourier series of the form

$$g(t) = a_0 + a_1 e^{it} + a_2 e^{2it} + \cdots .$$

By Lemma 8.5.13(ii), we have

$$\frac{\hat{\Lambda}(f)'(t)}{\hat{\Lambda}(f)(t)} = b_0 + b_1 e^{it} + b_2 e^{i2t} + \cdots ,$$

which implies

$$\frac{\hat{\Lambda}(f)'(t)}{\hat{\Lambda}(f)(t)} \cdot g(t) = c_0 + c_1 e^{it} + c_2 e^{i2t} + \cdots ,$$

i.e., $\hat{\Lambda}(f)'/\hat{\Lambda}(f) \cdot g \in \mathtt{F}^+$. Moreover, $\hat{\Lambda}(f) \in \mathtt{V}^E$ and therefore the map

$$(1 - \hat{\mathtt{P}}_{\hat{\Lambda}(f)}) : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$$

is a projection onto $\mathtt{F}^+ \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ due to Lemma 8.5.12. This implies

$$(1 - \hat{\mathtt{P}}_{\hat{\Lambda}(f)}).[\frac{\hat{\Lambda}(f)'}{\hat{\Lambda}(f)} \cdot g] = \frac{\hat{\Lambda}(f)'}{\hat{\Lambda}(f)} \cdot g.$$

Dividing the this equation by $\hat{\Lambda}(f)'/\hat{\Lambda}(f)$ results in

$$(1 - \hat{\mathtt{P}}_{\mathtt{H}_f}).g = g$$

and we conclude $\mathtt{F}^+ \subseteq (1 - \hat{\mathtt{P}}_{\mathtt{H}_f}).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$. $\square$

Lemma 8.5.15. *If $g \in \mathtt{V}^-$, then the map*

$$\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \lambda \mapsto g \cdot (\hat{\mathtt{P}}_g . \frac{\lambda}{g})$$

is a projection onto the subspace $\mathtt{F}^-$.

Proof. Define the function $b : \mathbb{S}^1 \to \mathbb{S}^1, \quad t \mapsto -t$, and let $f(t) = \frac{1}{g(-t)}$. By Lemma 2.4.10, we have $f \in \mathtt{V}^+$, and due to Proposition 8.5.12 the map

$$(1 - \hat{\mathtt{P}}_f) : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$$

is a projection onto $\mathtt{F}^+$. Hence, the map

$$\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad \lambda \mapsto ((1 - \hat{\mathtt{P}}_f).(\lambda \circ b)) \circ b$$

is a projection onto $\mathtt{F}^-$. The computation

$$\begin{aligned}
& \{[(1 - \hat{\mathtt{P}}_f).(\lambda \circ b)] \circ b\}(\theta) \\
= \;& [(1 - \hat{\mathtt{P}}_f).(\lambda \circ b)](-\theta) \\
\overset{8.5.1}{=} \;& [\lambda \circ b](-\theta) - \frac{i}{2\pi} \int_{\mathbb{S}^1} \frac{(\lambda \circ b)(t) - (\lambda \circ b)(-\theta)}{f(t) - f(-\theta)} \cdot f'(t) dt \\
= \;& \lambda(\theta) - \frac{i}{2\pi} \int_0^{2\pi} \frac{\lambda(\theta) - \lambda(-t)}{1/g(\theta) - 1/g(-t)} \cdot \frac{g'(-t)}{g^2(-t)} dt \\
= \;& \lambda(\theta) - \frac{i}{2\pi} \int_0^{2\pi} \frac{\lambda(\theta) \cdot -\lambda(t)}{1/g(\theta) - 1/g(t)} \cdot \frac{g'(t)}{g^2(t)} dt \\
\overset{2.4.3}{=} \;& \lambda(\theta) \cdot \frac{1}{2\pi i} \int_0^{2\pi} \frac{g'(t)}{g(t)} dt - \frac{i}{2\pi} \int_0^{2\pi} \frac{\lambda(\theta) - \lambda(t)}{1/g(\theta) - 1/g(t)} \cdot \frac{g'(t)}{g^2(t)} dt \\
= \;& \frac{1}{2\pi i} \int_0^{2\pi} \left[\frac{\lambda(\theta) \cdot g(t) - \lambda(\theta) \cdot g(\theta)}{g(t) - g(\theta)} + \frac{\lambda(\theta) \cdot g(\theta) - \lambda(t) \cdot g(\theta)}{g(t) - g(\theta)} \right] \frac{g'(t)}{g(t)} dt \\
= \;& \frac{1}{2\pi i} g(\theta) \cdot \int_0^{2\pi} \frac{\frac{\lambda(\theta)}{g(\theta)} - \frac{\lambda(t)}{g(t)}}{g(t) - g(\theta)} g'(t) dt \\
= \;& \frac{i}{2\pi} g(\theta) \cdot \int_0^{2\pi} \frac{\frac{\lambda(t)}{g(t)} - \frac{\lambda(\theta)}{g(\theta)}}{g(t) - g(\theta)} g'(t) dt \\
\overset{8.5.1}{=} \;& g(\theta) \cdot [\hat{\mathtt{P}}_g . \frac{\lambda}{g}](\theta)
\end{aligned}$$

shows

$$((1 - \hat{\mathtt{P}}_f).(\lambda \circ b)) \circ b = g \cdot (\hat{\mathtt{P}}_g . \frac{\lambda}{g}),$$

and we are finished. $\square$

Lemma 8.5.16 (Parameter invariance)**.** *Let* $f \in \mathtt{V}$*. If* $\lambda \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ *and* $\gamma \in \mathsf{Diff}^+\mathbb{S}^1$*, then*

$$[\hat{\mathtt{P}}_{f\circ\gamma}.(\lambda\circ\gamma)](\tau) = [\hat{\mathtt{P}}_f.\lambda](\gamma(\tau)\,)$$

for all $\tau \in \mathbb{S}^1$

Proof.

$$\begin{aligned}
[\hat{\mathtt{P}}_{f\circ\gamma}.(\lambda\circ\gamma)](\tau) &= \frac{i}{2\pi}\int_{\mathbb{S}^1} \frac{\lambda(\gamma(\theta)) - \lambda(\gamma(\tau))}{f(\gamma(\theta)) - f(\gamma(\tau))}\, f'(\gamma(\theta))\,\gamma'(\theta)\,d\theta \\
&= \frac{i}{2\pi}\int_{\mathbb{S}^1} \frac{\lambda(u) - \lambda(\gamma(\tau))}{f(u) - f(\gamma(\tau))}\, f'(u)\,du \\
&= (\hat{\mathtt{P}}_f.\lambda)(\gamma(\tau)).
\end{aligned}$$

$\square$

Lemma 8.5.17. *If* $f \in \mathtt{V}^-$*, then* $\hat{\mathtt{P}}_{\mathtt{H}_f}.\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) = \mathtt{H}_f$*.*

Proof.

$$\begin{aligned}
&\mathtt{H}_f \\
&\overset{8.4.1}{=} \{\, \hat{\mathtt{h}}_f.\lambda \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \,:\, \lambda \in \mathtt{F}^- \,\} \\
&\overset{8.5.15}{=} \{\, \hat{\mathtt{h}}_f.(f\cdot[\hat{\mathtt{P}}_f.\frac{\lambda}{f}]) \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \,:\, \lambda \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \,\} \\
&= \{\, \hat{\mathtt{h}}_f.(f\cdot[\hat{\mathtt{P}}_f.\lambda]) \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \,:\, \lambda \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \,\} \\
&\overset{8.4.1}{=} \{\, \frac{1}{\hat{\Lambda}(f)'}\cdot[(f\cdot[\hat{\mathtt{P}}_f.\lambda])\circ\hat{\Gamma}(f)^{-1}] \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \,:\, \lambda \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \,\} \\
&= \{\, \frac{(f\circ[\hat{\Gamma}(f)^{-1}])}{\hat{\Lambda}(f)'}\cdot[\hat{\mathtt{P}}_f.\lambda]\circ[\hat{\Gamma}(f)^{-1}] \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \,:\, \lambda \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \,\} \\
&\overset{8.5.16}{=} \{\, \frac{(f\circ[\hat{\Gamma}(f)^{-1}])}{\hat{\Lambda}(f)'}\cdot[\hat{\mathtt{P}}_{f\circ\hat{\Gamma}(f)^{-1}}.(\lambda\circ\hat{\Gamma}(f)^{-1})] \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \,:\, \lambda \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \,\} \\
&\overset{4.2.3(i)}{=} \{\, \frac{\hat{\Lambda}(f)}{\hat{\Lambda}(f)'}\cdot[\hat{\mathtt{P}}_{\hat{\Lambda}(f)}.(\lambda\circ\hat{\Gamma}(f)^{-1})] \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \,:\, \lambda \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \,\} \\
&= \{\, \frac{\hat{\Lambda}(f)}{\hat{\Lambda}(f)'}\cdot[\hat{\mathtt{P}}_{\hat{\Lambda}(f)}.\lambda] \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \,:\, \lambda \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \,\} \\
&\overset{8.5.2}{=} \hat{\mathtt{P}}_{\mathtt{H}_f}.\mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}).
\end{aligned}$$

$\square$

Lemma 8.5.18. *Let V be a vector space and $P_A, P_B : V \to V$ two projections. We define two subspaces $A := P_A.V$ and $B := P_B.V$. If*

$$(1 - P_A).V = (1 - P_B).V,$$

then the restriction of the projection

$$P_A|_B : B \to A$$

is the inverse function of

$$P_B|_A : A \to B.$$

Proof. Without loss of generality we only have to prove

$$P_A|_B \circ P_B|_A = \mathsf{id}_A,$$

because A and B are interchangeable in the formulas. The assumption

$$(1 - P_A).V = (1 - P_B).V$$

implies

$$(1 - P_A) \circ (1 - P_B) = (1 - P_B),$$

and hence $P_A \circ P_B = P_A$. $\square$

Proposition 8.5.19. *If $f \in \mathtt{V}$, then the map $\hat{\mathtt{P}}_K : \mathtt{H}_f \to \mathtt{K}$ is the inverse map of $\hat{\mathtt{P}}_{\mathtt{H}_f} : \mathtt{K} \to \mathtt{H}_f$.*

Proof. We will apply Lemma 8.5.18. The substitutions $V = \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$, $P_A = \hat{\mathtt{P}}_{\mathtt{H}_f}$ and $P_B = \hat{\mathtt{P}}_K$ yield

$$\begin{aligned} A &:= \hat{\mathtt{P}}_{\mathtt{H}_f}.\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \overset{8.5.17}{=} \mathtt{H}_f, \\ B &:= \hat{\mathtt{P}}_K.\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \overset{7.3.15(i)}{=} \hat{\mathtt{P}}_R(1 - \hat{\mathtt{P}}_0).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \overset{8.1.1}{=} \mathtt{K}. \end{aligned}$$

The hypothesis can be verified by

$$\begin{aligned} (1 - \hat{\mathtt{P}}_K).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) &\overset{7.3.15(ii)}{=} (\hat{\mathtt{P}}_0 + \hat{\mathtt{P}}_F).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \\ &\overset{7.3.10}{=} \mathtt{F}^+ \\ &\overset{8.5.14}{=} (1 - \hat{\mathtt{P}}_{\mathtt{H}_f}).\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \end{aligned}$$

and with the application of Lemma 8.5.18, we get the result. $\square$

8.6 Main Theorem

In Proposition 8.4.14, we have computed the derivative of $\hat{\Xi} : \mathtt{V}^- \to \mathtt{M}$ to be

$$d\hat{\Xi} : \mathtt{V}^- \times \mathtt{F}^- \to \mathtt{K}, \quad (f, \Delta\eta) \mapsto d\hat{\Xi}(f, \Delta\eta) = \hat{\omega}_{\hat{\Gamma}(f)^{-1}}.\hat{\mathtt{P}}_K.\hat{\mathtt{h}}_f.\Delta\eta.$$

In this section, we will invert it with respect to the right argument (see Lemma 8.6.9), in order to apply the Nash-Moser Theorem, which yields that $\hat{\Xi}$ is a tame diffeomorphism (see Theorem 8.6.7). Moreover, with a little calculation we will show that $\hat{\Xi}$ is holomorphic (see Proposition 8.6.8).

Lemma 8.6.1. *The inverse of*

$$\mathtt{V}^- \times \mathtt{K} \to \mathtt{K}, \quad (f, \lambda) \mapsto \hat{\omega}_{\hat{\Gamma}(f)^{-1}}.\lambda$$

with respect to the right argument reads

$$\mathtt{V}^- \times \mathtt{K} \to \mathtt{K}, \quad (f, \lambda) \mapsto \hat{\omega}_{\hat{\Gamma}(f)}.\lambda$$

and is smooth-tame.

Proof. The map

$$\hat{\Gamma} : \mathtt{V} \to \mathsf{Diff}^+\mathbb{S}^1$$

is smooth-tame by Lemma 7.6.5 and the inversion map

$$\iota : \mathsf{Diff}^+\mathbb{S}^1 \to \mathsf{Diff}^+\mathbb{S}^1$$

is smooth-tame by Lemma 5.2.13(ii). The composition map

$$\mathtt{C} : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \times \mathsf{Diff}^+\mathbb{S}^1 \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad (f, \gamma) \mapsto f \circ \gamma$$

is smooth-tame by Lemma 5.2.12 and the projection

$$(1 - \hat{\mathtt{P}}_0) : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$$

is smooth-tame by Lemma 7.3.5. Combining this with

$$\hat{\omega}_{\hat{\Gamma}(f)}.\lambda \overset{8.1.7}{=} (1 - \hat{\mathtt{P}}_0).\mathsf{R}_{\hat{\Gamma}(f)^{-1}}.\lambda = (1 - \hat{\mathtt{P}}_0).[\lambda \circ \iota(\hat{\Gamma}(f))].$$

we get the result. $\square$

Lemma 8.6.2. *The map*

$$\mathtt{V}^{-} \times \mathcal{C}^{\infty}(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^{\infty}(\mathbb{S}^1, \mathbb{C}), \quad (f, \lambda) \mapsto \hat{\mathtt{P}}_{\hat{\Lambda}(f)}.\lambda$$

is smooth-tame.

Proof. By Proposition 5.4.4 the map

$$\mathtt{V} \times \mathcal{C}^{\infty}(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^{\infty}(\mathbb{S}^1, \mathbb{C}), \quad (f, \lambda) \mapsto \hat{\mathtt{P}}_f.\lambda$$

with

$$(\hat{\mathtt{P}}_f.\lambda)(\tau) := \frac{i}{2\pi} \int_{\mathbb{S}^1} \frac{\lambda(\theta) - \lambda(\tau)}{f(\theta) - f(\tau)} f'(\theta) d\theta,$$

is smooth-tame, and by Lemma 7.6.4 the map $\hat{\Lambda} : \mathtt{V} \to \mathtt{V}^E$ is smooth-tame. Since by Lemma 5.2.8 the composition of smooth-tame maps is also smooth-tame the lemma is proved. ☐

Lemma 8.6.3. *The three maps*

(i) $\mathtt{V} \times \mathcal{C}^{\infty}(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^{\infty}(\mathbb{S}^1, \mathbb{C}), \quad (f, \lambda) \mapsto \dfrac{\hat{\Lambda}(f)}{\hat{\Lambda}(f)'} \cdot \lambda,$

(ii) $\mathtt{V} \times \mathcal{C}^{\infty}(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^{\infty}(\mathbb{S}^1, \mathbb{C}), \quad (f, \lambda) \mapsto \dfrac{\hat{\Lambda}(f)'}{\hat{\Lambda}(f)} \cdot \lambda,$ *and*

(iii) $\mathtt{V} \times \mathcal{C}^{\infty}(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^{\infty}(\mathbb{S}^1, \mathbb{C}), \quad (f, \lambda) \mapsto \hat{\Lambda}(f)' \cdot \lambda$

are smooth-tame.

Proof. The map

$$\mathtt{V} \to \mathtt{V}^E, \quad f \mapsto \hat{\Lambda}(f)$$

is smooth-tame by Lemma 7.6.4 and the derivative map is smooth-tame by Lemma 5.2.10(i), which implies that the function

$$\vartheta_1 : \mathtt{V} \to \mathcal{C}^{\infty}(\mathbb{S}^1, \mathbb{C}), \quad f \mapsto \hat{\Lambda}(f)'$$

is smooth-tame. Due to Lemma 5.3.6(iii), the two maps

$$\vartheta_2 : \mathtt{V} \to \mathcal{C}^{\infty}(\mathbb{S}^1, \mathbb{C}), \quad f \mapsto \frac{\hat{\Lambda}(f)}{\hat{\Lambda}(f)'}, \text{ and } \vartheta_3 : \mathtt{V} \to \mathcal{C}^{\infty}(\mathbb{S}^1, \mathbb{C}), \quad f \mapsto \frac{\hat{\Lambda}(f)'}{\hat{\Lambda}(f)}$$

are smooth-tame. In the three cases (i)-(iii) we have a smooth-tame map

$$\vartheta_j : \mathtt{V} \to \mathcal{C}^{\infty}(\mathbb{S}^1, \mathbb{C})$$

and conclude with Lemma 5.3.6(i) that

$$\mathtt{V} \times \mathcal{C}^{\infty}(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^{\infty}(\mathbb{S}^1, \mathbb{C}), \quad (f, \lambda) \mapsto \vartheta_j(f) \cdot \lambda$$

for $j = 1, 2, 3$ is smooth-tame. ☐

Lemma 8.6.4. *The inverse of*

$$\mathtt{V}^- \times \mathtt{H}_f \to \mathtt{K}, \quad (f,\lambda) \mapsto \hat{\mathtt{P}}_K.\lambda$$

with respect to the right argument reads

$$\mathtt{V}^- \times \mathtt{K} \to \mathtt{H}_f, \quad (f,\lambda) \mapsto \hat{\mathtt{P}}_{\mathtt{H}_f}.\lambda$$

and is smooth-tame.

Proof. By Proposition 8.5.19, the inverse of $\hat{\mathtt{P}}_K : \mathtt{H}_f \to \mathtt{K}$ is $\hat{\mathtt{P}}_{\mathtt{H}_f} : \mathtt{K} \to \mathtt{H}_f$. It remains to show the tameness. The expression

$$\hat{\mathtt{P}}_{\mathtt{H}_f}.\lambda = \frac{\hat{\Lambda}(f)}{\hat{\Lambda}(f)'} \cdot [\hat{\mathtt{P}}_{\hat{\Lambda}(f)}.\{\frac{\hat{\Lambda}(f)'}{\hat{\Lambda}(f)} \cdot \lambda\}]$$

is the composition of the following three functions, which are smooth-tame. The first function

$$\mathtt{V}^- \times \mathtt{K} \to \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}), \quad (f,\lambda) \mapsto \frac{\hat{\Lambda}(f)'}{\hat{\Lambda}(f)} \cdot \lambda$$

is smooth-tame by Lemma 8.6.3(ii). The second function

$$\mathtt{V}^- \times \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}), \quad (f,\lambda) \mapsto \hat{\mathtt{P}}_{\hat{\Lambda}(f)}.\lambda$$

is smooth-tame by Lemma 8.6.2 and the third function

$$\mathtt{V}^- \times \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{C}), \quad (f,\lambda) \mapsto \frac{\hat{\Lambda}(f)}{\hat{\Lambda}(f)'} \cdot \lambda$$

is smooth-tame by Lemma 8.6.3(i). The composition of these three maps is smooth-tame by Lemma 5.2.8. □

Lemma 8.6.5. *The inverse of*

$$\mathtt{V}^- \times \mathtt{F}^- \to \mathtt{H}_f, \quad (f,\lambda) \mapsto \hat{\mathtt{h}}_f.\lambda = \frac{1}{\hat{\Lambda}(f)'(\tau)} \cdot \lambda([\hat{\Gamma}(f)^{-1}](\tau))$$

with respect to the right argument reads

$$\mathtt{V}^- \times \mathtt{H}_f \to \mathtt{F}^-, \quad (f,\lambda) \mapsto \hat{\mathtt{h}}_f^{-1}.\lambda = [\hat{\Lambda}(f)' \cdot \lambda] \circ \hat{\Gamma}(f)$$

and is smooth-tame.

Proof. Recall, that by Definition 8.4.1 the function $\hat{\mathtt{h}}_f : \mathtt{F}^- \to \mathtt{H}_f$ is bijective for all $f \in \mathtt{V}$. The map

$$\mathtt{V} \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad (f, \lambda) \mapsto \hat{\Lambda}(f)' \cdot \lambda$$

is smooth-tame by Lemma 8.6.3(iii),

$$\hat{\Gamma} : \mathtt{V} \to \mathsf{Diff}^+\mathbb{S}^1$$

is smooth-tame by Lemma 7.6.5, and the composition map

$$\mathtt{C} : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}) \times \mathsf{Diff}^+\mathbb{S}^1 \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$$

is smooth-tame by Lemma 5.2.12. Hence, taking Lemma 5.2.8 into account, we conclude that the map

$$\mathtt{V}^- \times \mathtt{H}_f \to \mathtt{F}^-, \quad (f, \lambda) \mapsto [\hat{\Lambda}(f)' \cdot \lambda] \circ \hat{\Gamma}(f) = \mathtt{C}(\hat{\Lambda}(f)' \cdot \lambda, \hat{\Gamma}(f)\,).$$

□

Lemma 8.6.6. *Consider the function $\hat{\Xi} : \mathtt{V}^- \to \mathtt{M}$, and its derivative*

$$d\hat{\Xi} : \mathtt{V}^- \times \mathtt{F}^- \to \mathtt{K}, \quad (f, \Delta\eta) \mapsto d\hat{\Xi}(f, \Delta\eta) = \hat{\omega}_{\hat{\Gamma}(f)^{-1}}.\hat{\mathtt{P}}_K.\hat{\mathtt{h}}_f.\Delta\eta.$$

Then, the inverse with respect to the right argument reads

$$d\hat{\Xi}^\# : \mathtt{V}^- \times \mathtt{K} \to \mathtt{F}^-, \quad (f, b) \mapsto d\hat{\Xi}^\#(f, b) = \hat{\mathtt{h}}_f^{-1}.\hat{\mathtt{P}}_{\mathtt{H}_f}.\hat{\omega}_{\hat{\Gamma}(f)}.b$$

and is smooth-tame.

Proof. The function

$$\mathtt{V}^- \times \mathtt{K} \to \mathtt{K}, \quad (f, \lambda) \mapsto \hat{\omega}_{\hat{\Gamma}(f)}.\lambda$$

is smooth-tame by Lemma 8.6.1,

$$\mathtt{V}^- \times \mathtt{K} \to \mathtt{H}_f, \quad (f, \lambda) \mapsto \hat{\mathtt{P}}_{\mathtt{H}_f}.\lambda$$

is smooth-tame by Lemma 8.6.4, and

$$\mathtt{V}^- \times \mathtt{H}_f \to \mathtt{F}^-, \quad (f, \lambda) \mapsto \hat{\mathtt{h}}_f^{-1}.\lambda = [\hat{\Lambda}(f)' \cdot \lambda] \circ \hat{\Gamma}(f)$$

is smooth-tame by Lemma 8.6.5. This shows the lemma. □

Theorem 8.6.7. *The map $\hat{\Xi} : \mathtt{V}^- \to \mathtt{M}$ defined in 8.3.3 is a tame diffeomorphism.*

Proof. First we show that $\hat{\Xi}$ is smooth-tame. The function

$$\hat{\Pi} : \mathsf{Diff}^+\mathbb{S}^1 \to \mathtt{M}, \quad \gamma \mapsto (1 - \hat{\mathtt{P}}_0).(\gamma - \mathsf{id}_{\mathbb{S}^1})$$

is smooth by its definition and the map

$$\hat{\Gamma} : \mathtt{V} \to \mathsf{Diff}^+\mathbb{S}^1$$

is smooth-tame by Lemma 7.6.5. So far, we conclude that by Lemma 5.2.8 the composition $\hat{\Xi} = \hat{\Pi} \circ \hat{\Gamma}$ is smooth-tame. Furthermore, by Lemma 8.3.4 the function $\hat{\Xi} : \mathtt{V}^- \to \mathtt{M}$ is bijective, and by Lemma 8.6.6 the inverse of the derivative of $\hat{\Xi}$ with respect to the right argument is smooth-tame. Now all prerequisites of the Nash-Moser Theorem 5.2.15 are fulfilled, and we conclude that $\hat{\Xi}$ is a tame diffeomorphism. □

Proposition 8.6.8. *The function $\hat{\Xi} : \mathtt{V}^- \to \mathtt{M}$ is holomorphic. This reads in a formula*

$$d\hat{\Xi}(f, i\Delta f) = \hat{\mathtt{J}}_{\hat{\Xi}(f)}.d\hat{\Xi}(f, \Delta f)$$

with $f \in \mathtt{V}^-$ and $\Delta f \in \mathtt{F}^-$. Here, $\hat{\mathtt{J}}$ is the almost complex structure on $\mathtt{M}$ defined in 8.2.13.

Proof.

$$\begin{aligned}
d\hat{\Xi}(f, i\Delta f) &\overset{8.4.14}{=} \hat{\omega}_{\hat{\Gamma}(f)^{-1}}.\hat{\mathtt{P}}_K.\hat{\mathtt{h}}_f.\hat{\mathtt{I}}.\Delta f \\
&\overset{8.1.6}{=} \hat{\omega}_{\hat{\Gamma}(f)^{-1}}.\hat{\mathtt{P}}_K.\hat{\mathtt{I}}.\hat{\mathtt{h}}_f.\Delta f \\
&\overset{7.3.3(xii)}{=} -\hat{\omega}_{\hat{\Gamma}(f)^{-1}}.\hat{\mathtt{H}}.\hat{\mathtt{P}}_K.\hat{\mathtt{h}}_f.\Delta f \\
&\overset{8.1.7}{=} -\hat{\omega}_{\hat{\Gamma}(f)^{-1}}.\hat{\mathtt{H}}.\hat{\omega}_{\hat{\Gamma}(f)}.\hat{\omega}_{\hat{\Gamma}(f)^{-1}}.\hat{\mathtt{P}}_K.\hat{\mathtt{h}}_f.\Delta f \\
&\overset{8.4.14}{=} -\hat{\omega}_{\hat{\Gamma}(f)^{-1}}.\hat{\mathtt{H}}.\hat{\omega}_{\hat{\Gamma}(f)}.d\hat{\Xi}(f, \Delta f) \\
&\overset{8.2.13}{=} \hat{\mathtt{J}}_{\hat{\Pi}(\hat{\Gamma}(f))}.d\hat{\Xi}(f, \Delta f) \\
&\overset{8.3.3}{=} \hat{\mathtt{J}}_{\hat{\Xi}(f)}.d\hat{\Xi}(f, \Delta f).
\end{aligned}$$

□

Corollary 8.6.9. *The function*

$$\mathtt{V}^+ \to \mathtt{M}, \quad f \mapsto \hat{\Xi}(\hat{\Upsilon}(f))$$

is a tame diffeomorphism and is holomorphic.

Proof. The function

$$\hat{\Xi} : \mathtt{V}^- \to \mathtt{M}$$

is a tame diffeomorphism by Theorem 8.6.7 and holomorphic by Proposition 8.6.8. On the other hand, the function

$$\hat{\Upsilon} : \mathtt{V}^+ \to \mathtt{V}^-,$$

is bijective by Lemma 2.4.10. Taking Lemma 5.3.6 into account, we obtain that it is a tame diffeomorphism. Moreover, $\hat{\Upsilon}$ is holomorphic by Lemma 8.3.5. All this shows the lemma. □

Corollary 8.6.10. *The function* $\hat{\mathtt{J}} : \mathtt{M} \times \mathtt{K} \to \mathtt{K}$ *defined in Lemma 8.2.13 is a complex structure on the open subset* $\mathtt{M} \subseteq \mathtt{K}$*, which is considered as a Fréchet manifold. Moreover, every complex structure on* $\mathtt{M}$ *which is invariant under the action*

$$\hat{\rho} : \mathsf{Diff}^+\mathbb{S}^1 \times \mathtt{M} \to \mathtt{M}, \quad (\gamma, m) \mapsto \hat{\rho}_\gamma . m$$

defined in Lemma 8.1.9 is either equal to $\hat{\mathtt{J}}$ *or* $-\hat{\mathtt{J}}$.

Proof. The almost complex structure

$$\hat{\mathtt{J}} : \mathtt{M} \times \mathtt{K} \to \mathtt{K}, \quad (m, b) \mapsto \hat{\mathtt{J}}_m . b$$

defined in 8.2.13 is invariant under the action

$$\hat{\rho} : \mathsf{Diff}^+\mathbb{S}^1 \times \mathtt{M} \to \mathtt{M}.$$

The function $\hat{\Xi} : \mathtt{V}^- \to \mathtt{M}$ is a tame diffeomorphism by Theorem 8.6.7 and is holomorphic by Proposition 8.6.8. Therefore, $\hat{\mathtt{J}}$ as well as $-\hat{\mathtt{J}}$ is a complex structure on $\mathtt{M}$. By Lemma 8.2.6 every complex structure is an integrable almost complex structure. Moreover, Proposition 8.2.14 asserts that any almost complex structure on $\mathtt{M}$ which is invariant and integrable is equal to $\hat{\mathtt{J}}$ or equal to $-\hat{\mathtt{J}}$. Hence, we conclude every invariant complex structure on $\mathtt{M}$ is either equal to $\hat{\mathtt{J}}$ or equal to $-\hat{\mathtt{J}}$. □

Theorem 8.6.11 (Main Result)**.** *On the homogeneous space*

$$\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1,$$

there exists exactly one invariant complex structure up to sign.

Proof. Consider the open subset $\mathtt{M} \subseteq \mathtt{K}$ of the Fréchet space $\mathtt{K}$ defined in 8.1.1. By Lemma 8.1.6, the mapping

$$\hat{\Psi} : \mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1 \to \mathtt{M}, \quad \gamma \circ \mathsf{Rot}^+\mathbb{S}^1 \mapsto \hat{\Pi}(\gamma^{-1})$$

is well-defined and bijective. In this way, $\hat{\Psi}$ serves as a chart such that $\mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1$ becomes a Fréchet manifold. Moreover, by Lemma 8.1.9, the bijective function $\hat{\Psi} : \mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1 \to \mathtt{M}$ induces the affine action

$$\hat{\rho} : \mathsf{Diff}^+\mathbb{S}^1 \times \mathtt{M} \to \mathtt{M}, \quad (\gamma, m) \mapsto \hat{\rho}_\gamma.m.$$

Hence, Corollary 8.6.10 yields the result. $\square$

Chapter 9

Miscellaneous

This last chapter is independent of the others and consists of two parts. In the first three sections of this chapter, we will show the following: Consider an irreducible representation $\pi : A \to \mathsf{B}(H)$ of a C^*-algebra A on a Hilbert space H. Fix a natural number $n \in \mathbb{N}$. It turns out (9.3.4) that the unitary group of A acts transitively on $Gr_n(H)$, which is the set of all n-dimensional subspaces of H. In the second part (Section 4) of this chapter, we consider unitary representations of the restricted unitary group (see Definition 9.4.3) and compute the image of the momentum map.

Throughout this chapter, the hermitean product $\langle u|v\rangle$ on a Hilbert space H is $\mathbb{C}$-linear with respect to the right argument. Moreover, we use the ket-notation notation (see also 9.4.2) as follows. For $a, b \in H$ the the linear operator $|a\rangle\langle b| \in \mathsf{B}(H)$ is defined by

$$|a\rangle\langle b| : H \to H, \quad w \mapsto \langle b|w\rangle \cdot a.$$

9.1 Manifold Structure on $Gr_n(H)$

In this section, we define a manifold structure on the Grassmannian, the set of all n-dimensional subspaces of a given Hilbert space H, and discuss some properties.

Definition 9.1.1 (Tensors)**.** Let H be a Hilbert space and

$$\Lambda^n H = H \wedge H \wedge \cdots \wedge H$$

the n-fold antisymmetric tensor product with the Hilbert completion. For $\xi = (\xi_1, \ldots, \xi_n) \in H^n$ we write

$$\Lambda\xi := \xi_1 \wedge \cdots \wedge \xi_n \in \Lambda^n H$$

and

$$[\Lambda\xi] := \mathbb{C} \cdot \Lambda\xi \subseteq \Lambda^n H.$$

Definition 9.1.2. [Grassmannian] Let H be a Hilbert space. We define

$$Gr_n(H) := \{ [\Lambda\xi] \subseteq \Lambda^n H \,:\, (\xi_1, \ldots, \xi_n) \in H^n \wedge \Lambda\xi \neq 0 \}$$

to be the subset of the projective space of $\Lambda^n H$ consisting of all homogeneous tensors of rank n. This space is equivalent to the set of all vector subspaces of H of dimension n and therefore, we call it the *Grassmannian*.

Definition 9.1.3. For $\xi, w \in H^n$ we define the subspace

$$H_w^\perp := \{ x \in H \,:\, \langle x|w_j\rangle = 0, j = 1, \ldots, n \}$$

for $w = (w_1, \ldots, w_n) \in H^n$, and the matrix

$$A_{\xi w} := \begin{pmatrix} \langle w_1|\xi_1\rangle & \cdots & \langle w_n|\xi_1\rangle \\ \vdots & \ddots & \vdots \\ \langle w_1|\xi_n\rangle & \cdots & \langle w_n|\xi_n\rangle \end{pmatrix}.$$

Moreover, we define the subset

$$U_w := \{ [\Lambda\xi] \subseteq \Lambda^n H \,:\, \xi \in H^n \wedge \Lambda\xi \neq 0 \wedge \mathsf{det}\, A_{\xi w} \neq 0 \} \subseteq Gr_n(H).$$

Lemma 9.1.4 (Charts of $Gr_n(H)$)**.** *Consider an n-tuple*

$$w = (w_1, \ldots, w_n) \in H^n$$

with $\langle w_j|w_k\rangle = \delta_{jk}$ for $j, k = 1, \ldots, n$. Let

$$\Phi : (H_w^\perp)^n \to U_w \subseteq Gr_n(H), \quad x \mapsto [\Lambda(x + w)]$$

with

$$[\Lambda(x + w)] = [(x_1 + w_1) \wedge \cdots \wedge (x_n + w_n)]$$

be a chart of $Gr_n(H)$. Then Φ is injective and its inverse function is given by

$$\Phi^{-1} : U_w \to (H_w^\perp)^n, \quad [\Lambda\xi] \mapsto A_{\xi w}^{-1}.\xi - w.$$

Proof. 1. Step: $[\Lambda(x+w)] \in U_w$:
The vectors $w_1, \ldots, w_n \in H$ are linearly independent, and the vectors $x_1, \ldots, x_n \in H_w^\perp$ are orthogonal to all of them all. Therefore, $x_1+w_1, \ldots, x_n+w_n$ are linearly independent, so that $\Lambda(x+w) \neq 0$. Moreover, we have

$$\mathsf{det} A_{x+w,w} = \mathsf{det}(\underbrace{A_{xw}}_{=0} + \underbrace{A_{ww}}_{=\mathbb{1}}) \neq 0$$

and hence $[\Lambda(x+w)] \in U_w$ is shown.
2. Step: Φ is injective:
Let $x = (x_1, \cdots, x_n) \in (H_w^\perp)^n$ and define the function

$$\varphi_x : \langle w_1, \ldots, w_n \rangle_{\mathbb{C}} \to H_w^\perp$$

by $\varphi_x(w_1) = x_1, \ldots, \varphi_x(w_n) = x_n$. The graph of φ_x is the subspace

$$\langle w_1 + x_1, \ldots, w_n + x_n \rangle_{\mathbb{C}} \subseteq \langle w_1, \ldots, w_n \rangle_{\mathbb{C}} \oplus H_w^\perp,$$

and we conclude that Φ is injective.
3. Step: We will show that Φ^{-1} is well-defined:
Choose two elements $\xi, \eta \in H^n$ with $\Lambda\xi, \Lambda\eta \neq 0$ and $\mathsf{det}\, A_{\xi w}, \mathsf{det}\, A_{\eta w} \neq 0$ as representatives such that $[\Lambda\xi] = [\Lambda\eta] \in U_w$ holds. This implies that the n-tuples ξ and η generate the same n-dimensional vector subspace

$$V = \langle \xi_1, \ldots, \xi_n \rangle_{\mathbb{C}} = \langle \eta_1, \ldots, \eta_n \rangle_{\mathbb{C}}$$

of H. Hence, there exists an invertible matrix

$$T \in \mathsf{Mat}(n \times n, \mathbb{C})$$

with

$$\begin{aligned} T.\xi &= \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nn} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \\ &= \begin{pmatrix} T_{11}\xi_1 + \cdots + T_{1n}\xi_n \\ \vdots \\ T_{n1}\xi_1 + \cdots + T_{nn}\xi_n \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} = \eta. \end{aligned}$$

Using the definition of the matrix $A_{\xi w}$, we get

$$T \cdot A_{\xi w} = A_{\eta w},$$

which implies

$$A_{\xi w}^{-1}.\xi = (TA_{\xi w})^{-1}.T.\xi = A_{\eta w}^{-1}.\eta,$$

and hence

$$A_{\xi w}^{-1}.\xi - w = A_{\eta w}^{-1}.\eta - w.$$

This shows that Φ^{-1} is well-defined.
4. Step: We show that Φ^{-1} is the inverse function of Φ:
The composition

$$\Phi^{-1} \circ \Phi : (H_w^\perp)^n \to (H_w^\perp)^n, \quad x \mapsto A_{(x+w)\,w}^{-1} . (x+w) - w$$

is the identity map on $(H_w^\perp)^n$, because

$$A_{(x+w)\,w} = \underbrace{A_{xw}}_{=0} + \underbrace{A_{ww}}_{=\mathbb{1}} = \mathbb{1}$$

for $x \in H_w^\perp$. Conversely, we consider $[\Lambda\xi] \in U_w$, which means in particular $\mathsf{det}\, A_{\xi w}^{-1} \neq 0$. This implies that the vector subspace generated by $\xi = (\xi_1, \dots, \xi_n)$ is the same as the subspace generated by $A_{\xi w}^{-1}.\xi$. This is equivalent to

$$[\Lambda A_{\xi w}^{-1}.\xi] = [\Lambda \xi],$$

and we conclude that

$$\Phi \circ \Phi^{-1} : U_w \to U_w, \quad [\Lambda\xi] \mapsto [\Lambda(A_{\xi w}^{-1}.\xi - w + w)]$$

is the identity map on U_w. $\square$

Lemma 9.1.5. *If H is a Hilbert space, then the Grassmannian*

$$Gr_n(H) = \{\, [w_1 \wedge \cdots \wedge w_n] \subseteq \mathbb{P}(\Lambda^n H) \,:\, w_1, \dots, w_n \in H \,\}$$

is connected.

Proof. Let H_w and H_x be two n-dimensional vector subspaces of H, and

$$W = H_w + H_x \subseteq H$$

their sum. Let $m := \mathsf{dim}\, W$. Denote by $H_w^\perp$ and $H_x^\perp$ the Hilbert space complements of H_w and H_x in W respectively. Let $w_1, \dots, w_n$ be a basis of H_w, and $w_{k+1}, \dots, w_m$ a basis of the complement $H_w^\perp$, so that $w_1, \dots, w_m$ is a basis of W. Similarly, we define the basis $x_1, \dots, x_n$. With these data, we define a unitary mapping

$$u : W \to W, \quad y \mapsto w_1 \langle x_1 | y \rangle + \cdots + w_n \langle x_n | y \rangle$$

for all $y \in W$. It fulfills $u(H_x) = H_w$ and can be extended to H. Since the finite-dimensional Lie group $\mathsf{U}(W)$ is connected, there exists a path from H_w to H_x. $\square$

9.2 Transitivity Criterion

We consider a smooth action of a Banach–Lie group on a connected Banach manifold. In this section, we will provide a transitivity criterion based on the non-linear Open Mapping Theorem of Banach spaces. We need it for the next section.

Lemma 9.2.1. *Consider two Banach spaces V and W. Let*

$$B_r(x_0) = \{ \xi \in V \ : \ \|\xi - x_0\| < r \}$$

be the open ball around the point $x_0 \in V$ with radius $r \in \mathbb{R}^+$. We assume that the function

$$\Phi : B_r(x_0) \to W$$

fulfills the following properties:

(i) *Φ is smooth.*

(ii) *The derivative $d\Phi(x_0) : V \to W$ is surjective at $x_0 \in V$.*

(iii) *There exists a real number $k > 0$ such that*

$$\|d\Phi(\xi) - d\Phi(x_0)\| \leq k$$

holds for all $\xi \in B_r(x_0)$.

Then there exists a real number $\rho > 0$ such that

$$B_\rho(\Phi(x_0)\,) \subseteq \Phi(B_r(x_0)\,)$$

holds.

Proof. This follows from Corollary 15.2 on page 155 in [8], taking into account that smoothness implies weak G-differentiability. □

Lemma 9.2.2. *Let V and W be two Banach spaces, $U_V \subseteq V$ an open subset, and*

$$\Phi : U_V \to W$$

a smooth map. Fix a point $x_0 \in U_V$. If the derivative

$$d\Phi(x_0) : V \to W$$

at the point $x_0 \in U_V$ is surjective, then $\Phi(U_V)$ is a neighborhood of $\Phi(x_0)$.

Proof. The derivative

$$d\Phi : U_V \to \mathsf{B}(V, W)$$

is continuous, because Φ is smooth. Choose a real number $k > 0$. The subset $B_k(d\Phi(x_0)) \subseteq W$ is open, and its preimage is an open neighborhood of x_0. This implies that there exists an $r > 0$ such that

$$\|d\Phi(\xi) - d\Phi(x_0)\| \leq k$$

holds for all $\xi \in B_r(x_0)$. Choose r sufficiently small such that $B_r(x_0) \subseteq U_V$. Then all conditions of Lemma 9.2.1 are satisfied for the restriction of Φ to $B_r(x_0)$. This yields a real number $\rho > 0$ with

$$B_\rho(\Phi(x_0)) \subseteq \Phi(B_r(x_0)).$$

Hence, we have found the required open neighborhood $U_W := B_\rho(\Phi(x_0))$. □

Lemma 9.2.3 (Transitivity criterion)**.** *Let G be a Banach–Lie group, and*

$$\theta : G \times M \to M$$

a smooth action on a connected Banach manifold M. Let

$$\theta_p : G \to M, \quad a \mapsto a.p, \quad (a, p) \mapsto a.p$$

be the orbit map of $p \in M$. If for all $p \in M$ the derivative

$$d\theta_p(1) : T_1(G) \to T_p(M)$$

of the orbit map at $1 \in G$ is surjective, then G acts transitively on M.

Proof. Let $p \in M$ and consider the orbit map $\theta_p : G \to M$. By Lemma 9.2.2, there exists an open neighborhood $U_p \subseteq M$ of $p = \theta_p(1)$ such that

$$U_p \subseteq \theta_p(G).$$

This implies that every orbit is open. Furthermore, every orbit is closed, because it is the complement of the union of the remaining orbits. Since M is connected, M is the only orbit, i.e., the action is transitive. □

9.3 Transitivity Theorem

Let $\pi : A \to \mathsf{B}(H)$ be an irreducible unitary representation of a C^*-algebra A on the Hilbert space H and $n \in \mathbb{N}$. In this section, we will prove that the the unitary group

$$\mathsf{U}(A) := \{\, a \in A \,:\, a^* \cdot a = 1 = a \cdot a^* \,\}$$

acts transitively on the Grassmannian $Gr_n(H)$, the space of n-dimensional subspaces.

Lemma 9.3.1. *Let G be a Banach–Lie group and*

$$\pi : G \to \mathsf{U}(H), \quad a \mapsto \pi(a)$$

a unitary representation on the Hilbert space H which is continuous with respect to the operator norm. Let $w = (w_1, \ldots, w_n) \in H^n$ be chosen such that $\langle w_j | w_k \rangle = \delta_{jk}$ holds for $j, k = 1, \ldots, n$. Furthermore, let

$$\theta_w : G \to Gr_n(H), \quad a \mapsto a.[\Lambda w] = [\Lambda(\pi(a).w_1, \ldots, \pi(a).w_n)]$$

be the orbit mapping, and

$$\Phi_w : (H_w^\perp)^n \to U_w \subseteq Gr_n(H), \quad x \mapsto [\Lambda(x + w)]$$

the chart defined in Lemma 9.1.4, and chose an 1-neighborhood $G_w \subseteq G$ sufficientliy small. Then the derivative of

$$\Phi_w^{-1} \circ \theta_w : G_w \to (H_w^\perp)^n$$

at the point $1 \in G$ reads

$$\begin{aligned} d[\Phi_w^{-1} \circ \theta_w](1) : T_1(G) &\to (H_w^\perp)^n, \\ h &\mapsto (P_w^\perp.d\pi(h).w_1, \ldots, P_w^\perp.d\pi(h).w_n) \end{aligned}$$

with the projection

$$P_w^\perp := \mathsf{id}_H - (|w_1\rangle\langle w_1| + \cdots + |w_n\rangle\langle w_n|), \quad H \to H_w^\perp$$

and the derived representation

$$d\pi : T_1(G) \to \mathsf{gl}(H), \quad h \mapsto d\pi(h).$$

Proof. Using the explicit expression of Lemma 9.1.4,

$$\Phi^{-1} : U_w \to (H_w^{\perp})^n, \quad [\Lambda\xi] \mapsto A_{\xi w}^{-1}.\xi - w$$

and the definition of the orbit map θ_w we get

$$(\Phi_w^{-1} \circ \theta_w)(\mathsf{Exp}\, \epsilon \cdot h) = A_{\xi w}^{-1}.\xi - w \qquad (\dagger)$$

with

$$\xi = (\xi_1, \dots, \xi_n) = (\pi(\mathsf{Exp}\, \epsilon \cdot h).w_1, \dots, \pi(\mathsf{Exp}\, \epsilon \cdot h).w_n)$$

and

$$A = A_{\xi w} = \begin{pmatrix} \langle w_1 | \pi(\mathsf{Exp}\, \epsilon \cdot h).w_1 \rangle & \langle w_2 | \pi(\mathsf{Exp}\, \epsilon \cdot h).w_1 \rangle & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

The Taylor series of the matrix $A_{\xi w}$ with respect to ϵ is

$$A_{\xi w} = \mathbb{1}_n + \epsilon B + O(\epsilon^2)$$

with

$$B = \begin{pmatrix} \langle w_1 | d\pi(h).w_1 \rangle & \langle w_2 | d\pi(h).w_1 \rangle & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

and the Taylor series of $A_{\xi w}^{-1}$ is

$$A_{\xi w}^{-1} = \mathbb{1}_n - \epsilon B + O(\epsilon^2),$$

where $\mathbb{1}_n$ is the n-dimensional unit matrix. The Taylor series of the n-tuple ξ is

$$\xi = w + \epsilon\eta + O(\epsilon^2)$$

with

$$\eta = (d\pi(h).w_1, \dots, d\pi(h).w_n).$$

Substituting all this into equation (†) results in

$$\begin{aligned}(\Phi_w^{-1} \circ \theta_w)(\mathsf{Exp}\, \epsilon \cdot h) &= \big(\mathbb{1}_n - \epsilon B + O(\epsilon^2)\big) . \big(w + \epsilon\eta + O(\epsilon^2)\big) - w \\ &= \epsilon(\eta - B.w) + O(\epsilon^2).\end{aligned}$$

Now we compute the derivative

$$\begin{aligned}
\left(d(\Phi_w^{-1}\circ\theta_w)(1)\right)(h) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[(\Phi_w^{-1}\circ\theta_w)(\mathsf{Exp}\,\epsilon\cdot h)-\underbrace{(\Phi_w^{-1}\circ\theta_w)(1)}_{=0}\right]\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\epsilon(\eta-B.w)+O(\epsilon^2)-0\right]\\
&= \eta-B.w\\
&= (P_w^{\perp}.d\pi(h).w_1,\ldots,P_w^{\perp}.d\pi(h).w_n)
\end{aligned}$$

for all $h\in T_1(G)$. □

Lemma 9.3.2. *Let H be a Hilbert space and $A\subseteq \mathsf{B}(H)$ an involutive subalgebra which is dense with respect to the strong operator topology. Let*

$$\xi_1,\ldots,\xi_n\in H$$

be orthonormal vectors and

$$\eta_1,\ldots,\eta_n\in H$$

vectors with norm smaller than r. Assume there exists a hermitian operator $h\in\mathsf{B}(H)$ such that

$$h.\xi_1=\eta_1,\ldots,h.\xi_n=\eta_n.$$

Then there exists a hermitian element $b\in A$ which fulfills

$$b.\xi_1=\eta_1,\ldots,b.\xi_n=\eta_n.$$

Proof. This is a special case of Lemma 2.8.2 in [9] on page 43. □

Lemma 9.3.3. *Let A be a C^*-algebra, and*

$$\pi: A\to\mathsf{B}(H)$$

a representation on the Hilbert space H. If the image $\pi(A)\subseteq\mathsf{B}(H)$ is dense with respect to the strong operator topology, then the unitary group

$$\mathsf{U}(A):=\{\,a\in A\,:\,a^*\cdot a=1=a\cdot a^*\,\}$$

acts transitively on the Grassmannian

$$Gr_n(H):=\{\,[\Lambda\xi]\subseteq\Lambda^nH\,:\,\xi\in H^n\,,\,\Lambda\xi\neq 0\,\}\subseteq\mathbb{P}(\Lambda^nH).$$

Proof. Choose an n-tuple $w = (w_1, \ldots, w_n) \in H^n$ with $\langle w_j | w_k \rangle = \delta_{jk}$ as a representative. Let

$$\theta_w : \mathsf{U}(A) \to Gr_n(H), \quad a \mapsto a.[\Lambda w] = [\Lambda(\pi(a).w_1, \ldots, \pi(a).w_n)]$$

be the orbit map of $[\Lambda w] \in Gr_n(H)$, and

$$\Phi_w : H_w^\perp \to Gr_n(H), \quad x \mapsto [\Lambda(x + w)]$$

the chart of the Grassmannian $Gr_n(H)$ defined in Lemma 9.1.4. Denote by

$$d\pi : \mathsf{u}(A) \to \mathsf{B}(H), \quad \xi \mapsto d\pi(\xi)$$

the derived representation and by

$$\mathsf{u}(A) = \{ \xi \in A \, : \, \xi^* + \xi = 0 \}$$

the Lie algebra of $\mathsf{U}(A)$. Let $O_1 \subseteq \mathsf{U}(A)$ be a sufficiently small 1-neighborhood. By Lemma 9.3.1, the derivative of

$$\Phi_w^{-1} \circ \theta_w : O_1 \to (H_w^\perp)^n$$

at the point $1 \in \mathsf{U}(A)$ is

$$\begin{aligned} d(\Phi_w^{-1} \circ \theta_w)(1) : T_1(\mathsf{U}(A)) \to (H_w^\perp)^n, & \\ \xi \mapsto (P_w^\perp.d\pi(\xi).w_1, \ldots, P_w^\perp.d\pi(\xi).w_n), & \end{aligned}$$

and we will show that it is surjective. Fix an n-tuple $(x_1, \ldots, x_n) \in (H_w^\perp)^n$ which is a point of the image. The element

$$h := \sum_{j=1}^{n} |x_j\rangle\langle w_j| - |w_j\rangle\langle x_j| \in \mathsf{B}(H)$$

is anti-hermitian and satisfies $h.w_j = x_j$ for $j = 1, \ldots, n$. By Lemma 9.3.2, there exists an anti-hermitian element $b \in \pi(A)$ such that

$$b.w_j = x_j$$

holds for $j = 1, \ldots, n$, because the image $\pi(A) \subseteq \mathsf{B}(H)$ is dense with respect to the strong operator topology. This implies

$$(P_w^\perp.b.w_1, \ldots, P_w^\perp.b.w_n) = (x_1, \ldots, x_n),$$

due to $P_w^\perp.x_j = x_j$.

So far, we have shown that

$$d(\Phi_w^{-1} \circ \theta_w)(1) : \mathsf{u}(A) \to (H_w^{\perp})^n$$

is surjective. Taking into account that Φ_w is a chart, we see that the derivative

$$d\theta_w : \mathsf{T}(\mathsf{U}(A)) \to Gr_n(H)$$

of the orbit map is surjective at $1 \in \mathsf{U}(A)$. This is independent of the choice of

$$[\Lambda w] \in Gr_n(H).$$

Furthermore, by Lemma 9.1.5 the Grassmannian $Gr_n(H)$ is connected, and we can apply Lemma 9.2.3, which yields that $\mathsf{U}(A)$ acts transitively on $Gr_n(H)$. □

Theorem 9.3.4. *Let A be a C^*-algebra and*

$$\pi : A \to \mathsf{B}(H)$$

an irreducible representation on the Hilbert space H. Then the unitary group

$$\mathsf{U}(A) := \{ a \in A \, : \, a^* \cdot a = 1 = a \cdot a^* \}$$

acts transitively on the Grassmannian $Gr_n(H)$.

Proof. Since π is an irreducible representation, the only closed $\pi(A)$-invariant subspaces are $\{0\}$ and H. By Proposition 2.3.1 of [9] on page 29, we have

$$\pi(A)' := \{ a \in A \, : \, \forall_{\xi \in A}\, \xi \cdot a = a \cdot \xi \} = \mathbb{C} \cdot \mathsf{id}_H$$

for the commutant. This implies that $\pi(A) \subseteq \mathsf{B}(H)$ is dense with respect to the strong operator topology by the Theorem of von Neumann (see [2] on page 5). By Lemma 9.3.3, we conclude that the unitary group $\mathsf{U}(A)$ acts transitively on the Grassmannian $Gr_n(H)$. □

9.4 Restricted Unitary Group

Let H_0 be a Hilbert space and $\mathsf{U}_K(H_0)$ the group consisting of unitary operators $u \in \mathsf{U}_K(H_0)$ such that $u - \mathbb{1}$ is compact. We call $\mathsf{U}_K(H_0)$ the restricted unitary group. Let $\pi : \mathsf{U}_K(H_0) \to \mathsf{U}(V)$ be a norm-continuous unitary representation of the restricted unitary group on a Hilbert space V. Consider the momentum map

$$\Phi : \mathbb{P}(V) \to B_1(H_0), \quad [w] \mapsto \Phi([w])$$

defined by

$$\mathsf{tr}\,(\Phi([w]) \cdot \xi) = \frac{1}{i} \frac{\langle w | d\pi(\xi).w \rangle}{\langle w | w \rangle}.$$

In this section, we will show that the image of a coherent state vector w under the momentum map is a finite rank operator of the form

$$\Phi([w]) = \frac{1}{i} \sum_{k \in I} c_k |a_k\rangle\langle a_k|$$

with integer coefficients c_k.

Definition 9.4.1. Let H_0 be an infinite-dimensional Hilbert space. We write $\mathsf{B}(H_0)$ for the algebra of all bounded operators endowed with the norm topology. We write $\mathsf{K}(H_0)$ for the ideal consisting of compact operators. We write $\mathsf{U}(H_0)$ for the topological group of all unitary operator with the norm topology.

Definition 9.4.2 (Ket-bra notation)**.** Let $a, b \in H_0$ be two vectors, then we write $|a\rangle\langle b| \in \mathsf{B}(H_0)$ for the linear operator defined by

$$|a\rangle\langle b| : H_0 \to H_0, \quad w \mapsto \langle b|w\rangle \cdot a.$$

Where the scalar product $\langle b|w\rangle$ is $\mathbb{C}$-linear in w.

Remark: We have $|b\rangle\langle a| = (|a\rangle\langle b|)^*$.

Definition 9.4.3 (Restricted unitary group)**.** Let H_0 be an infinite-dimensional Hilbert space. We define the *restricted unitary group* $\mathsf{U}_K(H_0)$ to be the set of all unitary operators $u \in \mathsf{U}(H_0)$ such that $u - \mathbb{1}$ is compact, with the composition as the group multiplication. We use the topology which is induced by the operator norm.
Remark: We have to show that $\mathsf{U}_K(H_0)$ is closed under the group operations.

Proof. It can be seen that $\mathsf{U}_K(H_0)$ is closed under the multiplication in the following way. Let $u_1, u_2 \in \mathsf{U}_K(H_0)$, then

$$u_1 \cdot u_2 - \mathbb{1} = u_1 \cdot \underbrace{(u_2 - \mathbb{1})}_{\in \mathsf{K}(H_0)} + \underbrace{(u_1 - \mathbb{1})}_{\in \mathsf{K}(H_0)}$$

is compact, because $\mathsf{K}(H_0)$ is an ideal in $\mathsf{B}(H_0)$. The fact that $\mathsf{U}_K(H_0)$ is closed under the inverse can be obtained by

$$u^{-1} - \mathbb{1} = u^{-1}(1-u) \text{ for } u \in \mathsf{U}_K(H_0)$$

in a similar way. □

Remark: Be aware that the name "restricted unitary group" is often used otherwise.

Definition 9.4.4 (Restricted unitary Lie algebra)**.** For a given Hilbert space H_0 we define the *restricted unitary Lie algebra* $\mathsf{u}_K(H_0)$ to be the set of all anti-hermitian compact operator on H_0 with the commutator bracket as the Lie bracket. On the complexification

$$\mathsf{u}_K(H_0)_{\mathbb{C}} = \mathsf{u}_K(H_0) + i\mathsf{u}_K(H_0)$$

we define the involution

$$* : \mathsf{u}_K(H_0)_{\mathbb{C}} \to \mathsf{u}_K(H_0)_{\mathbb{C}}, \quad x \mapsto x^*$$

to be the adjunction, i.e., $< w|x.v > = < x^*.w|v >$ for all $w, v \in H_0$. We say $\mathsf{u}_K(H_0)_{\mathbb{C}} = \mathsf{K}(H_0)$ is the *complexified restricted unitary Lie algebra*.

Properties:

(i) $\xi^* = -\xi$ for all $\xi \in \mathsf{u}_K(H_0)$.

(ii) $\mathsf{u}_K(H_0)$ and $\mathsf{u}_K(H_0)_{\mathbb{C}}$ are closed subspaces of $\mathsf{B}(H_0)$.

(iii) The complexification of $\mathsf{u}_K(H_0)$ considered as a set is the set of all compact operators, i.e.,

$$\mathsf{u}_K(H_0)_{\mathbb{C}} = \mathsf{u}_K(H_0) + i\mathsf{u}_K(H_0) = K(H_0).$$

(iv) $\mathsf{u}_K(H_0)$ is the Lie algebra of $\mathsf{U}_K(H_0)$.

Definition 9.4.5. A *norm-continuous unitary representation of the restricted unitary group* on the Hilbert space V is a group homomorphism

$$\pi : \mathsf{U}_K(H_0)_{\mathbb{C}} \to \mathsf{U}(V)_{\mathbb{C}}, \quad a \mapsto \pi(a)$$

which is continuous with respect to the norm topology.

Lemma 9.4.6. *Consider the restricted unitary group* $\mathsf{U}_K(H_0)$*, and let*

$$\pi : \mathsf{U}_K(H_0) \to \mathsf{U}(V), \quad a \mapsto \pi(a)$$

be a norm-continuous unitary representation of the restricted unitary group. The derived representation

$$d\pi : \mathsf{u}_K(V) \to \mathsf{u}(V), \quad \xi \mapsto d\pi(\xi)$$

defined by

$$d\pi(\xi).w := \lim_{\epsilon\to 0} \frac{d}{d\epsilon}\left[\pi(\mathsf{Exp}\,\epsilon \cdot \xi).w\right]$$

fulfills the following properties:
(i) $d\pi : \mathsf{u}_K(H_0) \to \mathsf{u}(V)$ *is continuous, and*
(ii) $d\pi(\xi^*) = d\pi(\xi)^*$.

Proof. (i): By [4] (Theorem I in 8.1), any continuous group homomorphism between Banach–Lie groups is analytic. In particular, the morphism $d\pi : \mathsf{u}_K(H_0) \to \mathsf{u}(V)$ of Lie algebras corrrsponding to $\pi : \mathsf{U}_K(H_0) \to \mathsf{U}(V)$ is, like the derivative of π at the unit element, continuous.
(ii):

$$\begin{aligned}
\langle v|d\pi(\xi^*).w\rangle &= \langle v|\lim_{\epsilon\to 0}\frac{d}{d\epsilon}\left[\pi(\mathsf{Exp}\,\epsilon\cdot\xi^*).w\right]\rangle \\
&= \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\langle v|\pi(\mathsf{Exp}\,\epsilon\cdot\xi^*).w\rangle \\
&= \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\langle \pi(\mathsf{Exp}(-\epsilon\cdot\xi^*)).v|w\rangle \\
&= \lim_{\epsilon\to 0}\frac{d}{d\epsilon}\langle \pi(\mathsf{Exp}\,\epsilon\cdot\xi).v|w\rangle \\
&= \langle d\pi(\xi).v|w\rangle \\
&= \langle v|d\pi(\xi)^*.w\rangle.
\end{aligned}$$

□

Definition 9.4.7 (Coherent state orbit)**.** Consider the restricted unitary group $\mathsf{U}_K(H_0)$, and let

$$\pi : \mathsf{U}_K(H_0) \to \mathsf{U}(V), \quad a \mapsto \pi(a)$$

be a norm-continuous unitary representation with respect to the norm topology on $\mathsf{U}(V)$. A vector $w \in V$ is called *cyclic* if

$$\overline{d\pi(\mathcal{U}(\mathsf{u}_K(H_0))).w} = V,$$

where $\mathcal{U}(\mathsf{u}_K(H_0))$ is the universal enveloping algebra of $\mathsf{u}_K(H_0)$. A cyclic vector $w \in V$ is called a *coherent state vector*, if the orbit

$$\mathcal{O}_{[w]} \subseteq \mathbb{P}(V)$$

of $[w] = \mathbb{C} \cdot w \in \mathbb{P}(V)$ is a complex submanifold of $\mathbb{P}(V)$.

Remark: Then the tangent space

$$T_{[w]}(\mathcal{O}_{[w]}) \subseteq T_{[w]}(\mathbb{P}(V))$$

is a complex vector subspace, and $d\pi(\mathsf{u}_K(H_0)).w + \mathbb{C} \cdot w \subseteq V$ is also a complex vector subspace.

Definition 9.4.8 (Momentum map)**.** Let $\pi : \mathsf{U}_K(H_0) \to \mathsf{U}(V)$ be a norm-continuous unitary representation of the restricted unitary group $\mathsf{U}_K(H_0)$ on a Hilbert space V. Then the map

$$\phi : \mathbb{P}(V) \to B_1(H_0), \quad [w] \mapsto \phi([w])$$

defined by

$$\mathsf{tr}\,(\phi([w]) \cdot \xi) = \frac{1}{i} \frac{\langle w | d\pi(\xi).w \rangle}{\langle w | w \rangle}$$

for all $\xi \in g = \mathsf{u}_K(H_0)$ is called the momentum map.

Remark: The dual vector space of the Lie algebra $\mathsf{u}_K(H_0)$ is the vector space of anti-hermitian trace class operators, and the dual pairing is the trace of their product.

Lemma 9.4.9. *Let $\pi : \mathsf{U}_K(H_0) \to \mathsf{U}(V)$ be a norm-continuous unitary representation of the restricted unitary group, and $w \in V$. Then*

(i) $\phi([w])^* = -\phi([w])$, *i.e.*, $\phi([w]) \in \mathsf{u}_K(H_0)$, *and*

(ii) *there exists an orthonormal basis $\{a_j\}_{j \in I}$ and $c_j \in \mathbb{R}$ such that*

$$\phi([w]) = \frac{1}{i} \sum_{j \in I} c_j |a_j\rangle\langle a_j|.$$

Proof. (i):

$$\begin{aligned}
\mathsf{tr}(\phi([w])^* \cdot \xi) &= \overline{\mathsf{tr}(\phi([w]) \cdot \xi^*)} = -\frac{1}{i} \frac{\overline{\langle w | d\pi(\xi^*).w \rangle}}{\langle w | w \rangle} = -\frac{1}{i} \frac{\langle d\pi(\xi^*).w | w \rangle}{\langle w | w \rangle} \\
&= \frac{1}{i} \frac{\langle w | d\pi(\xi^*).w \rangle}{\langle w | w \rangle} = \mathsf{tr}(\phi([w]) \cdot \xi^*) \\
&= -\mathsf{tr}(\phi([w]) \cdot \xi)
\end{aligned}$$

for all $\xi \in \mathsf{u}_K(H_0)$.
(ii): This is the spectral decomposition taking into account that $\phi([w])$ is anti-hermitian and trace class. $\square$

In the rest of this section, we will show that the coefficients c_j are integers for a coherent state vector w.

Lemma 9.4.10. *Let $\pi : \mathsf{U}_K(H_0) \to \mathsf{U}(V)$ be a norm-continuous unitary representation of the restricted unitary group, and let $w \in V$ with $w \neq 0$ be a vector such that*

$$d\pi(\mathsf{u}_K(H_0)) + \mathbb{C} \cdot w \subseteq V$$

is a complex vector subspace. Let $\xi \in \mathsf{u}_K(H_0)$. If $[\phi([w]), \xi] = 0$, then $d\pi(\xi).w \in \mathbb{C} \cdot w$. Where $\phi([w]) \in B_1(H_0)$ and $[\cdot, \cdot]$ denotes the commutator.

Proof. We have $d\pi(\mathsf{u}_K(H_0)_{\mathbb{C}}) \cdot w + \mathbb{C} \cdot w = d\pi(\mathsf{u}_K(H_0)) \cdot w + \mathbb{C} \cdot w$, and therefore

$$d\pi(i\mathsf{u}_K(H_0)).w \subseteq d\pi(\mathsf{u}_K(H_0)).w + \mathbb{C} \cdot w.$$

Let $\xi \in \mathsf{u}_K(H_0)$. Then there exists an element $x \in \mathsf{u}_K(H_0)$ such that

$$d\pi(i\xi + x).w \in \mathbb{C} \cdot w.$$

Define $y := i\xi + x$ and observe

$$\xi = \frac{1}{2}(\xi - \xi^*) = -\frac{i}{2}(y - x + y^* - x^*) = -\frac{i}{2}(y + y^*).$$

Let

$$P_w = \frac{|w\rangle\langle w|}{\langle w|w\rangle}$$

be the orthogonal projection onto the subspace $\mathbb{C} \cdot w \in V$. We compute

$$\begin{aligned}
& \langle (\mathbb{1} - P_w).d\pi(y^*).w | (\mathbb{1} - P_w).d\pi(y^*).w\rangle \\
= \ & \langle w | d\pi(y)(\mathbb{1} - P_w) d\pi(y^*).w\rangle \\
= \ & \langle w | d\pi(y) d\pi(y^*) w\rangle - \langle w | d\pi(y) P_w d\pi(y^*) w\rangle \\
= \ & \langle w | d\pi(y) d\pi(y^*) w\rangle - \langle w | d\pi(y) w\rangle \frac{1}{\langle w|w\rangle} \langle w | d\pi(y^*) w\rangle \\
= \ & \langle w | d\pi(y) d\pi(y^*) w\rangle - \langle w | d\pi(y^*) w\rangle \frac{1}{\langle w|w\rangle} \langle w | d\pi(y) w\rangle \\
= \ & \langle w | d\pi(y) d\pi(y^*) w\rangle - \langle w | d\pi(y^*) P_w \underbrace{d\pi(y) w}_{\in \mathbb{C} \cdot w}\rangle \\
= \ & \langle w | d\pi(y) d\pi(y^*) w\rangle - \langle w | d\pi(y^*) d\pi(y) w\rangle \\
= \ & \langle w | d\pi([y, y^*]) w\rangle
\end{aligned}$$

$$\begin{aligned}
&= \langle w|d\pi([y,y^*])w\rangle \\
&= \langle w|d\pi([y,y+y^*])w\rangle \\
&= -2\langle w|w\rangle \frac{1}{i}\frac{\langle w|d\pi([y,\xi])w\rangle}{\langle w|w\rangle} \\
&= -2\langle w|w\rangle \mathsf{tr}(\phi([w])\cdot[y,\xi]) \\
&= -2\langle w|w\rangle \mathsf{tr}(\underbrace{[\xi,\phi([w])]}_{=0}\cdot y) \\
&= 0,
\end{aligned}$$

and get

$$(\mathbb{1}-P_w)d\pi(y^*).w = 0.$$

This implies $d\pi(y^*).w \in \mathbb{C}\cdot w$, and hence

$$d\pi(\xi).w = -\frac{i}{2}d\pi(y).w - \frac{i}{2}d\pi(y^*).w \in \mathbb{C}\cdot w$$

taking $\mathsf{ker}(1-P_w) = \mathbb{C}\cdot w$ into account. □

Remark:

The main part of the previous lemma comes from [23] and is adapted for our purposes. Moreover, this lemma has a geometric point of view; the restriction of the momentum map

$$\phi : \mathbb{P}(V) \to \mathfrak{g}^*, \quad [w] \mapsto \phi([w])$$

to the orbit of the coherent state vector is locally injective for the following reason. Write $\mathfrak{g}$ for the Lie algebra $\mathsf{u}_K(H_0)$. For $\eta \in \mathfrak{g}$, let $\theta_\eta \in \mathsf{Vec}(\mathcal{O}_{[w]})$ the corresponding vector field induced by the action. The derivative of the momentum map at $[w] \in \mathcal{O}_{[w]}$ is

$$(d\phi)_{[w]} : T_{[w]}(\mathcal{O}_w) \to \mathfrak{g}^*, \quad \theta_\eta([w]) \mapsto \{\xi \mapsto \frac{1}{i}\frac{\langle w|[d\pi(\xi), d\pi(\eta)].w\rangle}{\langle w|w\rangle}\}.$$

Chose $\eta \in \mathfrak{g}$ such that $(d\phi)_{[w]}(\theta_\eta([w]))$=0. This is equivalent to

$$\forall_{\xi\in\mathfrak{g}} \quad \frac{\langle w|[d\pi(\xi), d\pi(\eta)].w\rangle}{\langle w|w\rangle} = \mathsf{tr}(\phi([w])\cdot[\xi,\eta]) = 0,$$

which is equivalent to

$$[\phi([w]),\eta] = 0,$$

and Lemma 9.4.10 implies

$$d\pi(\eta).w \in \mathbb{C} \cdot w.$$

This means $\theta_\eta([w]) = 0$. So we observe that the derivative of the momentum map is injective at the point $[w] \in \mathcal{O}_{[w]}$, and we conclude that the momentum map is locally injective.

Proposition 9.4.11. *Let $\pi : \mathsf{U}_K(H_0) \to \mathsf{U}(V)$ be a norm-continuous unitary representation of the restricted unitary group, and let $w \in V$ with $w \neq 0$ be a coherent state vector. Then its image under the momentum map has the form*

$$\phi([w]) = \frac{1}{i} \sum_{k \in I} c_k |a_k\rangle\langle a_k|,$$

where $\{a_k\}_{k \in I} \subseteq V$ is an orthonormal basis, $c_j \in \mathbb{Z}$, and only finitely many coefficients are non-zero.

Proof. By Lemma 9.4.9, we have

$$\phi([w]) = \frac{1}{i} \sum_{k \in I} c_k |a_k\rangle\langle a_k|$$

with the orthonormal basis $\{a_k\}_{k \in I} \subseteq V$, and coefficients $c_k \in \mathbb{R}$. Next, we will show that the coefficients are integers. We have

$$[\, |a_j\rangle\langle a_j| \, , \, \phi([w]) \,] = 0.$$

Since $w \in V$ is a coherent state vector,

$$d\pi(\mathsf{u}_K(H_0)) + \mathbb{C} \cdot w \subseteq V$$

is a complex vector subspace and Lemma 9.4.10 yields

$$d\pi(|a_j\rangle\langle a_j|).w \in \mathbb{C} \cdot w,$$

which implies

$$d\pi(|a_j\rangle\langle a_j|).w = c_j \cdot w$$

due to

$$\frac{1}{i} c_j = \mathsf{tr}(\phi([w]) \cdot |a_j\rangle\langle a_j|) = \frac{1}{i} \frac{\langle w | d\pi(|a_j\rangle\langle a_j|).w\rangle}{\langle w|w\rangle}.$$

Since $|a_j\rangle\langle a_j| \in \mathsf{B}(H_0)$ is an orthogonal projection, we have

$$e^{i2\pi\cdot(|a_j\rangle\langle a_j|)} = \mathbb{1}.$$

Therefore,

$$w = \pi(e^{i2\pi\cdot(|a_j\rangle\langle a_j|)}).w = e^{i2\pi d\pi\cdot(|a_j\rangle\langle a_j|)}.w = e^{i2\pi c_j} \cdot w,$$

and hence $c_j \in \mathbb{Z}$. Since $\phi([w])$ is trace class, only a finite number of coefficients are non-zero. $\square$

Chapter 10

Notation

Common stuff

$\mathbb{1}$: the unit matrix
$f \equiv a$ means that the function f is the constant function.
$\mathbb{N} = \{1, 2, 3, \dots\}$
$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$
$\mathbb{R}^+ = \{\, x \in \mathbb{R} \,:\, x > 0 \,\}$
$\mathbb{R}_0^+ = \{\, x \in \mathbb{R} \,:\, x \geq 0 \,\}$
$\mathbb{C}^\times = \{\, z \in \mathbb{C} \,:\, z \neq 0 \,\}$
$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere
id_A identity map, on a set A
$F^\#$ inverse function with respect to the right argument (5.2.14)

Smooth functions

$\mathcal{C}^\infty(A, B)$: set of smooth functions (5.2.1)
L_ρ: left composition map (5.2.2)
R_ρ: right composition map (5.2.2)
$\mathbb{S}^1 := \mathbb{R}/2\pi\mathbb{Z}$ 1-sphere (2.1.1)
$\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ (2.1.2)
$\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$ (2.1.2)
$\mathsf{Diff}^+\mathbb{S}^1$: group of diffeomorphisms on the 1-sphere (2.1.5)
$\mathsf{id}_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1, \quad t \mapsto t$ (2.1.5)
$\iota : \mathsf{Diff}^+\mathbb{S}^1 \to \mathsf{Diff}^+\mathbb{S}^1, \quad \gamma \mapsto \gamma^{-1}$ (2.1.5)
C: composition map

Subsets and functions of complex numbers

$\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$ (2.3.1)
$\overline{\mathbb{D}} := \{ z \in \mathbb{C} : |z| \leq 1 \}$ (2.3.1)
$\partial\mathbb{D} := \{ z \in \mathbb{C} : |z| = 1 \}$ (2.3.1)
$\mathtt{D}_f$ interior of $f(\mathbb{S}^1)$ (2.4.4)
$\overline{\mathtt{D}}_f$ (2.4.4)
$\partial\mathtt{D}_f := f(\mathbb{S}^1)$ (2.4.4)
$\mathtt{r}_f$: tube radius (6.1.1)
$\mathtt{N}_f(\xi, t) := f(t) - i\frac{f'(t)}{|f'(t)|} \cdot \xi$ (6.1.3)
$\mathtt{A}_f := \mathtt{N}_f((-\mathtt{r}_f, \mathtt{r}_f) \times \mathbb{S}^1) \subseteq \mathbb{C}$ tube neighborhood of $f(\mathbb{S}^1)$ (6.1.3)
$\pi_1 : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}$ projection (6.1.9)
$\pi_2 : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{S}^1$ projection (6.1.9)

Subsets of $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$

$\mathtt{F}^+ := \{a_0 + a_1 e^{it} + a_2 e^{2it} + \cdots\}$ (2.2.3)
$\mathtt{F}^- := \{a_0 + a_1 e^{-it} + a_2 e^{-2it} + \cdots\}$ (2.2.3)
$\mathtt{E}_0 := \{r + b_1 e^{it} + b_2 e^{2it} + \cdots\}$ (7.3.11)
$\mathtt{E}_1 := \{re^{it} + b_1 e^{2it} + b_2 e^{3it} + \cdots\}$ (7.2.1)
$\mathtt{K} := \{\cdots + \overline{b_2} e^{-2it} + \overline{b_1} e^{-it} + b_1 e^{it} + b_2 e^{2it} + \cdots\}$ (8.1.1)
$\mathtt{W} \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times)$ and $\mathtt{w}(f) = 0$ (7.2.3)
$\mathtt{V} \subseteq \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times)$ diffeomorphic embeddings with $\mathtt{w}(f) = 1$ (2.4.3)
$\mathtt{V}^E := \mathtt{V} \cap \mathtt{E}_1$ and $r > 0$ (2.4.3)
$\mathtt{V}^- := \{e^{it}(1 + b_1 e^{-it} + b_2 e^{-2it} + \cdots)\}$ (2.4.3)
$\mathtt{V}^+ := \{e^{it}(1 + a_1 e^{it} + a_2 e^{2it} + \cdots)\}$ (2.4.3)
$\mathtt{U}_f \subseteq \mathtt{V}$: Open neighborhood of f (7.1.1)
$\mathtt{O}_f \subseteq \mathtt{V}$: Smaller open neighborhood of f (7.4.4)
$\mathtt{O}_f^E \subseteq \mathtt{V}^E$: Restriction of $\mathtt{O}_f$ to an affine hyper plane (7.4.4)
$\mathtt{M} = \{ m \in \mathtt{K} : m' < -1 \}$ (8.1.2)

Linear functions on subsets of $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$

$\hat{\mathtt{P}}_R$: real part of a function (7.3.2, 7.3.6)
$\hat{\mathtt{P}}_I := 1 - \hat{\mathtt{P}}_R$ (7.3.2, 7.3.6)
$\hat{\mathtt{P}}_0 \lambda = \frac{1}{2\pi} \int_{\mathbb{S}^1} \lambda(t) dt$ (7.3.2, 7.3.6)
$\hat{\mathtt{P}}_F = \frac{1}{2}(1 - \hat{\mathtt{P}}_0) - \frac{1}{2}\hat{\mathtt{I}}\hat{\mathtt{H}}$ (7.3.2, 7.3.8)
$\hat{\mathtt{P}}_K = 2\hat{\mathtt{P}}_R(1 - \hat{\mathtt{P}}_0 - \hat{\mathtt{P}}_F)$ (7.3.2)
$\hat{\mathtt{I}}$: multiplication with i (7.3.2, 7.3.6)
$(\hat{\mathtt{H}}.\lambda)(\tau) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \frac{\lambda(\tau) - \lambda(\theta)}{\tan\frac{\tau - \theta}{2}} d\theta$ (7.3.2, 7.3.13)

Nonlinear function on subsets of $\mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$

$\hat{\Upsilon} : \mathtt{V} \to \mathtt{V}$ with $\hat{\Upsilon}(f)(t) = 1/f(-t)$ (8.3.5)
$\hat{\Lambda} : \mathtt{V} \to \mathtt{V}^E$ (4.2.2)
$\hat{\Gamma} : \mathtt{V} \to \mathsf{Diff}^+\mathbb{S}^1$ (4.2.2)
$\hat{\Theta}_f : \mathtt{U}_f \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}), \quad \eta \mapsto (\pi_1 \circ \mathtt{N}_f^{-1} \circ \eta) \circ (\pi_2 \circ \mathtt{N}_f^{-1} \circ \eta)^{-1}$ (7.1.3)
$\hat{\Sigma}_f : \mathtt{U}_f \to \mathsf{Diff}^+\mathbb{S}^1, \quad \eta \mapsto \pi_2 \circ \mathtt{N}_f^{-1} \circ \eta$ (7.1.3)
$\hat{\theta}_f = \hat{\Theta}_f|_{\mathtt{V}^E \cap \mathtt{U}_f}$ (7.1.4)
$\hat{k}_f : \mathtt{U}_f \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C})$ (7.2.5)
$\mathtt{w} : \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}^\times) \to \mathbb{Z}$: winding number (2.4.1)
$\hat{l} : \mathtt{W} \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{C}), \quad f \mapsto \hat{l}(f) = (1 - \hat{\mathtt{P}}_0).\{t \mapsto \int_0^t \frac{f'(\theta)}{f(\theta)} d\theta\}$ (7.4.1)
$\hat{\mathtt{m}} : \mathtt{W} \to \mathbb{C}$ with $\hat{\mathtt{m}}(f) := f(\tau) \cdot e^{-\hat{l}(f)(\tau)}$ (7.4.2)
$\hat{\delta}_f : \mathtt{O}_f^E \times \mathtt{E}_1 \to \mathtt{E}_0$ (7.5.1)
$\hat{\vartheta}_f : \mathtt{O}_f^E \times \mathtt{E}_0 \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$ (7.5.1)
$\hat{\sigma}_f : \mathtt{O}_f^E \times \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}) \to \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$ (7.5.1)

Chapter 8

$\hat{\Pi} : \mathsf{Diff}^+\mathbb{S}^1 \to \mathtt{M}, \quad \gamma \mapsto \gamma - \hat{\mathtt{P}}_0.(\gamma - \mathsf{id}_{\mathbb{S}^1})$ (8.1.3)
$\hat{\Psi} : \mathsf{Diff}^+\mathbb{S}^1/\mathsf{Rot}^+\mathbb{S}^1 \to \mathtt{M}, \quad \gamma \mapsto \gamma - \hat{\mathtt{P}}_0.(\gamma - \mathsf{id}_{\mathbb{S}^1})$ (8.1.6)
$\hat{\Xi} : \mathtt{V}^- \to \mathtt{M}, \quad \eta \mapsto \hat{\Pi}(\hat{\Gamma}(\eta))$ (8.3.3)
$\hat{\mathtt{J}} : \mathtt{M} \times \mathtt{K} \to \mathtt{K}$: complex structure on $\mathtt{M}$ (7.4.1)
$\hat{\rho} : \mathsf{Diff}^+\mathbb{S}^1 \times \mathtt{M} \to \mathtt{M}$ action of $\mathsf{Diff}^+\mathbb{S}^1$ on $\mathtt{M}$ (7.4.6)
$\hat{\omega}_\gamma : \mathtt{K} \to \mathtt{K}, \quad \lambda \mapsto (1 - \hat{\mathtt{P}}_0).[\lambda \circ \gamma^{-1}]$ (8.1.7)
$\hat{\mathtt{h}}_f : \mathtt{F}^- \to \mathtt{H}_f$ with $\hat{\mathtt{h}}_f(\lambda) = \frac{1}{\hat{\Lambda}(f)'} \cdot [\lambda \circ \hat{\Gamma}(f)^{-1}]$ (8.4.1)
$\mathtt{H}_f := \hat{\mathtt{h}}_f(\mathtt{F}^-)$ (8.4.1)
$\hat{\mathtt{P}}_f.\lambda := \frac{i}{2\pi} \int_{\mathbb{S}^1} \frac{\lambda(\theta) - \lambda(\tau)}{f(\theta) - f(\tau)} f'(\theta) d\theta$ (8.5.1)
$(\hat{\mathtt{P}}_{\mathtt{H}_f}.\lambda).(t) = \frac{\hat{\Lambda}(f)(t)}{\hat{\Lambda}(f)'(t)} [\hat{\mathtt{P}}_{\hat{\Lambda}(f)}.\{\theta \mapsto \lambda(\theta) \frac{\hat{\Lambda}(f)'(\theta)}{\hat{\Lambda}(f)(\theta)}\}](t)$ (8.5.2)
$[\hat{\Omega}_f.\lambda](z) := \frac{1}{2\pi i} \int_0^{2\pi} \frac{\lambda(t)}{f(t) - z} \cdot f'(t) dt$ (8.5.3)
$\hat{\Omega}_E = \hat{\Omega}_f$ with $f : t \mapsto e^{it}$ (8.5.3)

Chapter 9

$Gr_n(H)$ (9.1.2)
$\mathsf{B}(H)$: bounded operators on H
$\mathsf{B}_1(H)$: trace class operators on H
$\mathsf{K}(H)$: compact operators on H
$\mathsf{U}(H)$: unitary operators on H
$\mathsf{u}(H)$: anti-hermitian operators on H

$\mathsf{U}_K(H)$: restricted unitary group (9.4.3)
$\mathsf{u}_K(H)$: restricted unitary Lie algebra (9.4.4)
$|a\rangle\langle b|$: (9.4.2)
$\mathcal{U}$: universal enveloping algebra

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